Introduction

The following thesis plays a central role in deformation theory:

$(*)$ If X is a moduli space over a field k of characteristic zero, then a formal neighborhood of any point $x \in X$ is controlled by a differential graded Lie algebra.

This idea was developed in unpublished work of Deligne, Drinfeld, and Feigin, and has powerfully influenced subsequent contributions of Hinich, Kontsevich-Soibelman, Manetti, and many others. The goal of this paper is to give a precise formulation of $(*)$ using the language of higher category theory. Our main result is Theorem 6.20, which can be regarded as an analogue of $(*)$ in the setting of noncommutative geometry. Our proof uses a method which can be adapted to prove a version of $(*)$ itself (Theorem 5.3).

Let us now outline the contents of this paper. Our first step is to define precisely what we mean by a moduli space. We will adopt Grothendieck's "functor of points" philosophy: giving the moduli space X is equivalent to specifying the functor $R \mapsto X(R) = \mathrm{Hom}(\operatorname{Spec} R, X)$. We will consider several variations on this theme:

(a) Allowing R to range over the category Ring of commutative rings, we obtain the notion of a *classical moduli problem* (Definition 1.3). We will discuss this notion and give several examples in §1.

(b) To understand the deformation theory of a moduli space X, it is often useful to extend the definition of the functor $R \mapsto X(R)$ to a more general class of rings. Algebraic topology provides such a generalization via the theory of *E_∞-ring spectra* (or, as we will call them, *E_∞-rings*). We will review this theory in §3 and use it to formulate the notion of a *derived moduli problem* (Definition 3.3).

(c) Let k be a field. To study the local structure of a moduli space X near a point $x \in X(k)$, it is useful to restrict our attention to the values $X(R)$ where R is a ring which is, in some sense, very similar to k (for example, local Artin algebras having residue field k). In §4, we will make this precise by introducing the notion of a *formal moduli problem* (Definition 4.6).

(d) Another way of enlarging the category of commutative rings is by weakening the requirement of commutativity. In the setting of ring spectra there are several flavors of commutativity available, given by the theory of *E_n-rings* for $0 \leq n < \infty$. We will review the theory of E_n-rings in §6, and use it to formulate the notion of a *formal E_n-moduli problem*.

In order to adequately treat cases (b) through (d), it is important to note that for $0 \leq n \leq \infty$, an E_n-ring is an essentially homotopy-theoretic object, and should therefore be treated using the formalism of higher category theory. In §2 we will give an overview of this formalism; in particular, we introduce the notion of an *∞-category* (Definition 2.9). Most of the basic objects under consideration in this paper form ∞-categories, and our main results can be formulated as equivalences of ∞-categories:

$(*')$ If k is a field of characteristic zero, then the ∞-category of formal moduli problems over k is equivalent to the ∞-category of differential graded Lie algebras over k (Theorem 5.3).

$(*'')$ If k is any field and $0 \leq n < \infty$, the ∞-category of formal E_n moduli problems over k is equivalent to the ∞-category of augmented E_n-algebras over k (Theorem 6.20).

We will formulate these statements more precisely in §5 and §6, respectively.

Assertions $(*')$ and $(*'')$ can be regarded as instances of *Koszul duality*: $(*')$ reflects a duality between commutative and Lie algebras, while $(*'')$ reflects a duality of the theory of E_n-algebras with itself. We will investigate this second duality in §7 by introducing a contravariant functor $A \mapsto \mathbb{D}(A)$ from the ∞-category of augmented E_n-algebras to itself. In §8, we will explain how to use this duality functor to construct the equivalence $(*'')$.

The remaining sections of this paper are devoted to examples. In §9, we will describe how the ideas of this paper can be applied to study the deformation theory of (differential graded) categories. In §10, we give a very brief description of ongoing joint work with Dennis Gaitsgory, which describes the braided monoidal deformations of the representation category of a reductive algebraic group.

Remark 0.1. The subject of deformation theory has a voluminous literature, some of which has substantial overlap with the material discussed in this paper. Though we have tried to provide relevant references in the body of the text, there are undoubtedly many sins of omission for which we apologize in advance.

Warning 0.2. The approach to the study of deformation theory described in this paper makes extensive use of higher category theory. We will sketch some of the central ideas of this theory in §2, and then proceed to use these ideas in an informal way. For a more comprehensive approach, we refer the reader to the author's book [22]. All of the unproven assertions made in §1 through §8 of this paper, with the exception properties $(K1)$ through $(K3)$ of the Koszul duality functor (see §7), can be found in the book [22] or the series of papers [23].

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1. Moduli Problems for Commutative Rings

Let Ring denote the category of commutative rings and $\mathcal{S}\mathrm{et}$ the category of sets. Throughout this paper, we will make extensive use of Grothendieck's "functor of points" philosophy: that is, we will identify a geometric object X (such as a scheme) with the functor $\mathrm{Ring} \to \mathcal{S}\mathrm{et}$ represented by X, given by the formula $R \mapsto \mathrm{Hom}(\operatorname{Spec} R, X)$.

Example 1.1. Let $X : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ be the functor which assigns to each commutative ring R the set $R^{\times}$ of invertible elements of R. For any commutative ring R, we have a canonical bijection $X(R) = R^{\times} \simeq \mathrm{Hom}_{\mathrm{Ring}}(\mathbf{Z}[t^{\pm 1}], R)$. In other words, we can identify X with the functor represented by the commutative ring $\mathbf{Z}[t^{\pm 1}]$.

Example 1.2. Fix an integer $n \geq 0$. We define a functor $X : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ by letting $F(R)$ denote the set of all submodules $M \subseteq R^{n+1}$ such that the quotient R^{n+1}/M is a projective R-module of rank n (from which it follows that M is a projective R-module of rank 1). The functor X is not representable by a commutative ring. However, it is representable in the larger category Sch of *schemes*. That is, for any commutative ring R we have a canonical bijection $X(R) \simeq \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, \mathbb{P}^n)$, where $\mathbb{P}^n \simeq \operatorname{Proj} \mathbf{Z}[x_0, \ldots, x_n]$ denotes projective space of dimension n.

For some purposes, the notion of a functor $X : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ is too restrictive. We often want to study moduli problems X which assign to a commutative ring R some class of geometric objects which depend on R. The trouble is that this collection of geometric objects is naturally organized into a category, rather than a set. This motivates the following definition:

Definition 1.3. Let Gpd denote the collection of *groupoids*: that is, categories in which every morphism is an isomorphism. We regard Gpd as a 2-category: morphisms are given by functors between groupoids, and 2-morphisms are given by natural transformations (which are automatically invertible). A *classical moduli problem* is a functor $X : \mathrm{Ring} \to \mathrm{Gpd}$.

Remark 1.4. Every set S can be regarded as a groupoid by setting

$$\mathrm{Hom}_S(x, y) = \begin{cases} \{\mathrm{id}_x\} & \text{if } x = y \\ \emptyset & \text{if } x \neq y. \end{cases}$$

This construction allows us to identify the category $\mathcal{S}\mathrm{et}$ with a full subcategory of the 2-category Gpd. In particular, every functor $X : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ can be identified with a classical moduli problem in the sense of Definition 1.3.

Example 1.5. For every commutative ring R, let $X(R)$ be the category of elliptic curves $E \to \operatorname{Spec} R$ (morphisms in the category $X(R)$ are given by isomorphisms of elliptic curves). Then F determines a functor $\mathrm{Ring} \to \mathrm{Gpd}$, and can therefore be regarded as a moduli problem in the sense of Definition 1.3. This moduli problem cannot be represented by a commutative ring or even by a scheme: for any scheme Y, $\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, Y)$ is a set. In particular, if we regard $\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, Y)$ as a groupoid, every object has a trivial automorphism group. In contrast, every object of $X(R)$ has a *nontrivial* automorphism group: every elliptic curve admits a nontrivial automorphism, given by multiplication by -1.

Nevertheless, the moduli problem X is representable if we work not in the category of schemes but in the larger 2-category $\mathrm{St}_{\mathrm{DM}}$ of *Deligne-Mumford stacks*. More precisely, there exists a Deligne-Mumford stack $\mathcal{M}_{\mathrm{Ell}}$ (the *moduli stack of elliptic curves*) for which there is a canonical equivalence of categories $X(R) \simeq \mathrm{Hom}_{\mathrm{St}_{\mathrm{DM}}}(\operatorname{Spec} R, \mathcal{M}_{\mathrm{Ell}})$ for every commutative ring R.

Example 1.6. Fix an integer $n \geq 0$. For every commutative ring R, let $X(R)$ denote the category whose objects are projective R-modules of rank n, and whose morphisms are given by isomorphisms of R-modules. Then X can be regarded as a moduli problem Ring $\rightarrow$ Gpd. This moduli problem is not representable in the 2-category $\mathrm{St}_{\mathrm{DM}}$ of Deligne-Mumford stacks, because projective R-modules admit continuous families of automorphisms. However, F is representable in the larger 2-category $\mathrm{St}_{\mathrm{Art}}$ of *Artin stacks*. Namely, there is an Artin stack $\mathrm{BGL}(n) \in \mathrm{St}_{\mathrm{Art}}$ for which there is a canonical bijection $X(R) \simeq \mathrm{Hom}_{\mathrm{St}_{\mathrm{Art}}}(\operatorname{Spec} R, \mathrm{BGL}(n))$ for every commutative ring R.

2. Higher Category Theory

In §1, we discussed the notion of a moduli problem in classical algebraic geometry. Even very simple moduli problems involve the classification of geometric objects which admit nontrivial automorphisms, and should therefore be treated as categories rather than as sets (Examples 1.5 and 1.6). Consequently, moduli problems themselves (and the geometric objects which represent them) are organized not into a category, but into a 2-category. Our discussion in this paper will take us much further into the realm of higher categories. We will devote this section to providing an informal overview of the ideas involved.

Definition 2.1 (Informal)**.** Let $n \geq 0$ be a nonnegative integer. The notion of an *n-category* is defined by induction on n. If $n = 0$, an n-category is simply a set. If $n > 0$, an n-category $\mathcal{C}$ consists of the following:

1. A collection of objects $X, Y, Z, \ldots$
2. For every pair of objects $X, Y \in \mathcal{C}$, an $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
3. Composition laws $\phi_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$ which are required to be unital and associative.

If η is an object of the $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for some pair of objects $X, Y \in \mathcal{C}$, then we will say that η is a 1-*morphism of* $\mathcal{C}$. More generally, a *k-morphism* in $\mathcal{C}$ is a $(k-1)$-morphism in some $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Example 2.2. Every topological space X determines an n-category $\pi_{\leq n} X$, the *fundamental n-groupoid of* X. If $n = 0$, we let $\pi_{\leq n} X = \pi_0 X$ be the set of path components of X. For $n > 0$, we let $\pi_{\leq n} X$ be the n-category whose objects are points of X, where $\mathrm{Hom}_{\pi_{\leq n}}(x, y)$ is the fundamental $(n-1)$-groupoid $\pi_{\leq n-1} P_{x,y}(X)$, where $P_{x,y}(X) = \{p : [0,1] \rightarrow X : p(0) = x, p(1) = y\}$ is the space of paths from x to y in X. Composition in $\pi_{\leq n} X$ is given by concatenation of paths. If $n = 1$, this definition recovers the usual fundamental groupoid of X.

Definition 2.1 is informal because we did not specify precisely what sort of associative law the composition in $\mathcal{C}$ is required to satisfy. If $n = 1$, there is no real ambiguity and Definition 2.1 recovers the usual definition of a category. When $n = 2$, the situation is more subtle: the associative law should posit the commutativity of a diagram having the form

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(W, X) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) & \xrightarrow{\phi_{W,X,Y}} & \mathrm{Hom}_{\mathcal{C}}(W, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \\ \downarrow{\scriptstyle \phi_{X,Y,Z}} & & \downarrow{\scriptstyle \phi_{W,Y,Z}} \\ \mathrm{Hom}_{\mathcal{C}}(W, X) \times \mathrm{Hom}_{\mathcal{C}}(X, Z) & \xrightarrow{\phi_{W,X,Z}} & \mathrm{Hom}_{\mathcal{C}}(W, Z). \end{array}$$

Since this is a diagram of categories and functors, rather than sets and functions, we are faced with a question: do we require this diagram to commute "on the nose" or only up to isomorphism? In the former case, we obtain the definition of a *strict* 2-*category*. This generalizes in a straightforward way: we can require strict associativity in Definition 2.1 to obtain a notion of strict n-category for every n. However, this notion turns out to be of limited use. For example, the fundamental n-groupoid of a topological space $\pi_{\leq n}X$ usually cannot be realized as a strict n-category when $n > 2$.

To accommodate Example 2.2, it is necessary to interpret Definition 2.1 differently. In place of equality, we require the existence of *isomorphisms*

$$\gamma_{W,X,Y,Z} : \phi_{W,X,Z} \circ (\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(W,X)} \times \phi_{X,Y,Z}) \simeq \phi_{W,Y,Z} \circ (\phi_{W,X,Y} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)}).$$

These isomorphisms are themselves part of the structure of $\mathcal{C}$, and are required to satisfy certain coherence conditions. When $n > 2$, these coherence conditions are themselves only required to hold up to isomorphism: *these* isomorphisms must also be specified and required to satisfy further coherences, and so forth. As n grows, it becomes prohibitively difficult to specify these coherences explicitly.

The situation is dramatically simpler if we wish to study not arbitrary n-categories, but *n-groupoids*. An n-category $\mathcal{C}$ is called an n-groupoid if every k-morphism in $\mathcal{C}$ is invertible. If X is any topological space, then the n-category $\pi_{\leq n}X$ is an example of an n-groupoid: for example, the 1-morphisms in $\pi_{\leq n}X$ are given by paths $p : [0, 1] \to X$, and every path p has an inverse q (up to homotopy) given by $q(t) = p(1 - t)$. In fact, all n-groupoids arise in this way. To formulate this more precisely, let us recall that a topological space X is an *n-type* if the homotopy groups $\pi_m(X, x)$ are trivial for every $m > n$ and every point $x \in X$. The following idea goes back (at least) to Grothendieck:

Thesis 2.3. *The construction $X \mapsto \pi_{\leq n}X$ establishes a bijective correspondence between n-types (up to weak homotopy equivalence) and n-groupoids (up to equivalence).*

We call Thesis 2.3 a thesis, rather than a theorem, because we have not given a precise definition of n-categories (or n-groupoids) in this paper. Thesis 2.3 should instead be regarded as a requirement that any reasonable definition of n-category must satisfy: when we restrict to n-categories where all morphisms are invertible, we should recover the usual homotopy theory of n-types. On the other hand, it is easy to concoct a definition of n-groupoid which tautologically satisfies this requirement:

Definition 2.4. An *n-groupoid* is an n-type.

Definition 2.4 has an evident extension to the case $n = \infty$:

Definition 2.5. An *∞-groupoid* is a topological space.

It is possible to make sense of Definition 2.1 also in the case where $n = \infty$: that is, we can talk about higher categories which have k-morphisms for every positive integer k. In the case where all of these morphisms turn out to be invertible, this reduces to the classical homotopy theory of topological spaces. We will be interested in the next simplest case:

Definition 2.6 (Informal). An *$(\infty, 1)$-category* is an ∞-category in which every k-morphism is invertible for $k > 1$.

In other words, an $(\infty, 1)$-category $\mathcal{C}$ consists of a collection of objects together with an ∞-groupoid $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for every pair of objects $X, Y \in \mathcal{C}$, which are equipped with an associative composition law. We can therefore use Definition 2.5 to formulate a more precise version of Definition 2.6.

Definition 2.7. A *topological category* is a category $\mathcal{C}$ for which each of the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with a topology, and each of the compositions maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is continuous. If $\mathcal{C}$ and $\mathcal{D}$ are topological categories, we will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *continuous* if, for every pair of objects $X, Y \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is continuous. The collection of (small) topological categories and continuous functors forms a category, which we will denote by $\mathcal{C}\mathrm{at}_t$.

Construction 2.8. Let $\mathcal{C}$ be a topological category. We can associate to $\mathcal{C}$ an ordinary category $\mathrm{h}\mathcal{C}$ as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y)$: that is, maps from X to Y in $\mathrm{h}\mathcal{C}$ are homotopy classes of maps from X to Y in $\mathcal{C}$.

We say that a morphism f in $\mathcal{C}$ is an *equivalence* if the image of f in $h\mathcal{C}$ is an isomorphism.

Definition 2.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a continuous functor between topological categories. We will say that F is a *weak equivalence* if the following conditions are satisfied:

(1) The functor F induces an equivalence of ordinary categories $h\mathcal{C} \to h\mathcal{D}$.
(2) For every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$$

is a weak homotopy equivalence.

Let $h\mathcal{C}\mathrm{at}_\infty$ be the category obtained from $\mathcal{C}\mathrm{at}_t$ by formally inverting the collection of weak equivalences. An $(\infty, 1)$*-category* is an object of $h\mathcal{C}\mathrm{at}_\infty$. We will refer to $h\mathcal{C}\mathrm{at}_\infty$ as the *homotopy category of* $(\infty, 1)$*-categories*.

Remark 2.10. More precisely, we should say that $h\mathcal{C}\mathrm{at}_\infty$ is the homotopy category of *small* $(\infty, 1)$-categories. We will also consider $(\infty, 1)$-categories which are not small.

Remark 2.11. There are numerous approaches to the theory of $(\infty, 1)$-categories which are now known to be equivalent, in the sense that they generate categories equivalent to $h\mathcal{C}\mathrm{at}_\infty$. The approach described above (based on Definitions 2.7 and 2.9) is probably the easiest to grasp psychologically, but is one of the most difficult to actually work with. We refer the reader to [1] for a description of some alternatives to Definition 2.7 and their relationship to one another.

All of the higher categories we consider in this paper will have k-morphisms invertible for $k > 1$. Consequently, it will be convenient for us to adopt the following:

Convention 2.12. The term ∞*-category* will refer to an $(\infty, 1)$-category $\mathcal{C}$ in the sense of Definition 2.9. That is, we will implicitly assume that all k-morphisms in $\mathcal{C}$ are invertible for $k > 1$.

With some effort, one can show that Definition 2.7 gives rise to a rich and powerful theory of ∞-categories, which admits generalizations of most of the important ideas from classical category theory. For example, one can develop ∞-categorical analogues of the theories of limits, colimits, adjoint functors, sheaves, and so forth. Throughout this paper, we will make free use of these ideas; for details, we refer the reader to [22].

Example 2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Then there exists another ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with the following universal property: for every ∞-category $\mathcal{C}'$, there is a canonical bijection

$$\mathrm{Hom}_{h\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}', \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Hom}_{h\mathcal{C}\mathrm{at}_\infty}(\mathcal{C} \times \mathcal{C}', \mathcal{D}).$$

We will refer to objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ simply as *functors* from $\mathcal{C}$ to $\mathcal{D}$.

Warning 2.14. By definition, an ∞-category $\mathcal{C}$ is simply an object of $h\mathcal{C}\mathrm{at}_\infty$: that is, a topological category. However, there are generally objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ which are not given by continuous functors between the underlying topological categories.

Warning 2.15. The process of generalizing from the setting of ordinary categories to the setting of ∞-categories is not always straightforward. For example, if $\mathcal{C}$ is an ordinary category, then a *product* of a pair of objects X and Y is another object Z equipped with a pair of maps $X \leftarrow Z \to Y$ having the following property: for every object $C \in \mathcal{C}$, the induced map $\theta : \mathrm{Hom}_{\mathcal{C}}(C, Z) \to \mathrm{Hom}_{\mathcal{C}}(C, X) \times \mathrm{Hom}_{\mathcal{C}}(C, Y)$ is a bijection. In the ∞-categorical context, it is natural to demand not that θ is bijective but instead that it is a weak homotopy equivalence. Consequently, products in $\mathcal{C}$ viewed as an ordinary category (enriched over topological spaces) are not necessarily the same as products in $\mathcal{C}$ when viewed as an ∞-category. To avoid confusion, limits and colimits in the ∞-category $\mathcal{C}$ are sometimes called *homotopy limits* and *homotopy colimits*.

We close this section by describing a method which can be used to construct a large class of examples of ∞-categories.

Construction 2.16. Let $\mathcal{C}$ be an ordinary category and let W be a collection of morphisms in $\mathcal{C}$. Then we let $\mathcal{C}[W^{-1}]$ denote an ∞-category which is equipped with a functor $\alpha : \mathcal{C} \to \mathcal{C}[W^{-1}]$ having the following universal property: for every ∞-category $\mathcal{D}$, composition with α induces a fully faithful embedding

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

whose essential image consists of those functors which carry every morphism in W to an equivalence in $\mathcal{D}$. More informally: $\mathcal{C}[W^{-1}]$ is the ∞-category obtained from $\mathcal{C}$ by formally inverting the morphisms in W.

Example 2.17. Let $\mathcal{C}$ be the category of all topological spaces and let W be the collection of weak homotopy equivalences. We will refer to $\mathcal{C}[W^{-1}]$ as the *∞-category of spaces*, and denote it by $\mathcal{S}$.

Example 2.18. Let R be an associative ring and let Chain_R denote the category of chain complexes of R-modules. A morphism $f : M_\bullet \to N_\bullet$ in Chain_R is said to be a *quasi-isomorphism* if the induced map of homology groups $\mathrm{H}_n(M) \to \mathrm{H}_n(N)$ is an isomorphism for every integer n. Let W be the collection of all quasi-isomorphisms in $\mathcal{C}$; then $\mathrm{Chain}_R[W^{-1}]$ is an ∞-category which we will denote by Mod_R. The homotopy category hMod_R can be identified with the classical *derived category of R-modules.*

Example 2.19. Let k be a field of characteristic zero. A *differential graded Lie algebra* over k is a Lie algebra object of the category Chain_k: that is, a chain complex of k-vector spaces $\mathfrak{g}_\bullet$ equipped with a Lie bracket operation $[,] : \mathfrak{g}_\bullet \otimes \mathfrak{g}_\bullet \to \mathfrak{g}_\bullet$ which satisfies the identities

$$[x, y] + (-1)^{d(x)d(y)}[y, x] = 0$$

$$(-1)^{d(z)d(x)}[x, [y, z]] + (-1)^{d(x)d(y)}[y, [z, x]] + (-1)^{d(y)d(z)}[z, [x, y]] = 0$$

for homogeneous elements $x \in \mathfrak{g}_{d(x)}$, $y \in \mathfrak{g}_{d(y)}$, $z \in \mathfrak{g}_{d(z)}$. Let $\mathcal{C}$ be the category of differential graded Lie algebras over k and let W be the collection of morphisms in $\mathcal{C}$ which induce a quasi-isomorphism between the underlying chain complexes. Then $\mathcal{C}[W^{-1}]$ is an ∞-category which we will denote by $\mathrm{Lie}_k^{\mathrm{dg}}$; we will refer to $\mathrm{Lie}_k^{\mathrm{dg}}$ as the *∞-category of differential graded Lie algebras over k.*

Example 2.20. Let $\mathcal{C}\mathrm{at}_t$ be the ordinary category of Definition 2.9, whose objects are topologically enriched categories and whose morphisms are continuous functors. Let W be the collection of all weak equivalences in $\mathcal{C}\mathrm{at}_t$ and set $\mathcal{C}\mathrm{at}_\infty = \mathcal{C}\mathrm{at}_t[W^{-1}]$. We will refer to $\mathcal{C}\mathrm{at}_\infty$ as the *∞-category of (small) ∞-categories*. By construction, the homotopy category of $\mathcal{C}\mathrm{at}_\infty$ is equivalent to the category $\mathrm{h}\mathcal{C}\mathrm{at}_\infty$ of Definition 2.9.

3. Higher Algebra

Arguably the most important example of an ∞-category is the ∞-category $\mathcal{S}$ of *spaces* of Example 2.17. A more explicit description of $\mathcal{S}$ can be given as follows:

(*a*) The objects of $\mathcal{S}$ are CW complexes.
(*b*) For every pair of CW complexes X and Y, we let $\mathrm{Hom}_{\mathcal{S}}(X, Y)$ denote the space of continuous maps from X to Y (endowed with the compact-open topology).

The role of $\mathcal{S}$ in the theory of ∞-categories is analogous to that of the ordinary category of sets in classical category theory. For example, for any ∞-category $\mathcal{C}$ one can define a *Yoneda embedding* $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, given by $j(C)(D) = \mathrm{Hom}_{\mathcal{C}}(D, C) \in \mathcal{S}$.

In this paper, we will be interested in studying the ∞-categorical analogues of more algebraic structures like commutative rings. As a first step, we recall the following notion from stable homotopy theory:

Definition 3.1. A *spectrum* is a sequence of pointed spaces $X_0, X_1, \ldots \in \mathcal{S}_*$ equipped with weak homotopy equivalences $X_n \simeq \Omega X_{n+1}$; here $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ denotes the based loop space functor $X \mapsto \{p : [0,1] \to X | p(0) = p(1) = *\}$.

To any spectrum X, we can associate abelian groups $\pi_k X$ for every integer k, defined by $\pi_k X = \pi_{k+n} X_n$ for $n \gg 0$. We say that X is *connective* if $\pi_n X \simeq 0$ for $n < 0$.

The collection of spectra is itself organized into an ∞-category which we will denote by Sp. If $X = \{X_n, \alpha_n : X_n \simeq \Omega X_{n+1}\}_{n \geq 0}$ is a spectrum, then we will refer to X_0 as the *0th space* of X. The construction $X \mapsto X_0$ determines a forgetful functor $\mathrm{Sp} \to \mathcal{S}$, which we will denote by Ω^∞.

We will say a spectrum X is *discrete* if the homotopy groups $\pi_i X$ vanish for $i \neq 0$. The construction $X \mapsto \pi_0 X$ determines an equivalence from the ∞-category of discrete spectra to the ordinary category of abelian groups. In other words, we can regard the ∞-category Sp as an *enlargement* of the ordinary category of abelian groups, just as the ∞-category $\mathcal{S}$ is an enlargment of the ordinary category of sets.

The category Ab of abelian groups is an example of a *symmetric monoidal* category: that is, there is a tensor product operation $\otimes : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ which is commutative and associative up to isomorphism. This operation has a counterpart in the setting of spectra: namely, the ∞-category Sp admits a symmetric monoidal structure $\wedge : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. This operation is called the *smash product*, and is compatible with the usual tensor product of abelian groups in the following sense: if X and Y are connective spectra, then there is a canonical isomorphism of abelian groups $\pi_0(X \wedge Y) \simeq \pi_0 X \otimes \pi_0 Y$. The unit object for the the smash product $\wedge$ is called the *sphere spectrum* and denoted by S.

The symmetric monoidal structure on the ∞-category Sp allows us to define an ∞-category CAlg(Sp) of *commutative algebra objects* of Sp. An object of CAlg(Sp) is a spectrum R equipped with a multiplication $R \wedge R \to R$ which is unital, associative, and commutative up to coherent homotopy. We will refer to the objects of CAlg(Sp) as *E_∞-rings*, and to CAlg(Sp) as the *∞-category of E_∞-rings*. The sphere spectrum S can be regarded as an E_∞-ring in an essentially unique way, and is an initial object of the ∞-category CAlg(Sp).

For any E_∞-ring R, the product on R determines a multiplication on the direct sum $\pi_* R = \oplus_n \pi_n R$. This multiplication is unital, associative, and commutative in the graded sense (that is, for $x \in \pi_i R$ and $y \in \pi_j R$ we have $xy = (-1)^{ij} yx \in \pi_{i+j}(R)$). In particular, $\pi_0 R$ is a commutative ring and each $\pi_i R$ has the structure of a module over $\pi_0 R$. The construction $R \mapsto \pi_0 R$ determines an equivalence between the ∞-category of *discrete* E_∞-rings and the ordinary category of commutative rings. Consequently, we can view CAlg(Sp) as an enlargement of the ordinary category of commutative rings.

Remark 3.2. To every E_∞-ring R, we can associate an ∞-category $\mathrm{Mod}_R(\mathrm{Sp})$ of *R-module spectra*: that is, modules over R in the ∞-category of spectra. If M and N are R-module spectra, we will denote the space $\mathrm{Hom}_{\mathrm{Mod}_R(\mathrm{Sp})}(M,N)$ simply by $\mathrm{Hom}_R(M,N)$. If M is an R-module spectrum, then $\pi_* M$ is a graded module over the ring $\pi_* R$. In particular, each homotopy group $\pi_n M$ has the structure of a $\pi_0 R$-module. If R is a discrete commutative ring, then $\mathrm{Mod}_R(\mathrm{Sp})$ can be identified with the ∞-category $\mathrm{Mod}_R = \mathrm{Chain}_R[W^{-1}]$ of Example 2.18. In particular, the homotopy category $\mathrm{hMod}_R(\mathrm{Sp})$ is equivalent to the classical derived category of R-modules.

We have the following table of analogies:

Classical Notion	∞-Categorical Analogue
Set	topological space
Category	∞-Category
Abelian group	Spectrum
Commutative Ring	E_∞-Ring
Ring of integers $\mathbf{Z}$	Sphere spectrum S

Motivated by these analogies, we introduce the following variant Definition 1.3:

Definition 3.3. A *derived moduli problem* is a functor X from the ∞-category CAlg(Sp) of E_∞-rings to the ∞-category $\mathcal{S}$ of spaces.

Remark 3.4. Suppose that $X_0 : \mathrm{Ring} \to \mathrm{Gpd}$ is a classical moduli problem. We will say that a derived moduli problem $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ is an *enhancement of F* if, whenever R is a commutative ring (regarded as a discrete E_∞-ring), we have an equivalence of categories $X_0(R) \simeq \pi_{\leq 1} X(R)$, and the homotopy groups $\pi_i X(R)$ vanish for $i \geq 2$ (and any choice of base point).

Example 3.5. Let A be an E_∞-ring. Then R defines a derived moduli problem, given by the formula $X(R) = \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Sp})}(A, R)$. Assume that A is connective: that is, the homotopy groups $\pi_i A$ vanish for $i < 0$. Then X can be regarded as an enhancement of the classical moduli problem $\mathrm{Spec}(\pi_0 A) : R \mapsto \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 A, R)$.

Example 3.6. Let R be an E_∞-ring and let M be an R-module spectrum. We say that M is *projective of rank* r if the $\pi_0 R$-module $\pi_0 M$ is projective of rank r, and the map $\pi_k R \otimes_{\pi_0 R} \pi_0 M \to \pi_k M$ is an isomorphism for every integer k. Fix an integer $n \geq 0$. For every E_∞-ring R, let $X(R)$ denote the ∞-category space for maps of R-modules $f : M \to R^{n+1}$ such that the cofiber R^{n+1}/M is a projective R-module of rank n; the maps in $X(R)$ are given by homotopy equivalences of R-modules (compatible with the map to R^{n+1}). The $X(R)$ is an ∞-groupoid, so we can regard X as a functor $\mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$. Then X is a derived moduli problem, which is an enhancement of the classical moduli problem represented by the scheme $\mathbb{P}^n = \mathrm{Proj}\, \mathbf{Z}[x_0, x_1, \ldots, x_n]$ (Example 1.2). We can think of X as providing a generalization of projective space to the setting of E_∞-rings.

Example 3.7. Let X be the functor which associates to every E_∞-ring R the ∞-groupoid of projective R-modules of rank n. Then $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ is a derived moduli problem, which can be regarded as an enhancement of the classical moduli problem represented by the Artin stack $\mathrm{BGL}(n)$ (Example 1.6).

Let us now summarize several motivations for the study of derived moduli problems:

(*a*) Let $\mathfrak{X}_0$ be a scheme (or, more generally, an algebraic stack), and let X_0 be the classical moduli problem given by the formula $F_0(R) = \mathrm{Hom}(\mathrm{Spec}\, R, \mathfrak{X}_0)$. Examples 3.6 and 3.7 illustrate the following general phenomenon: we can often give a conceptual description of $X_0(R)$ which continues to make sense in the case where R is an arbitrary E_∞-ring, and thereby obtain a derived moduli problem $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ which enhances X_0. In these cases, one can often think of X as itself being represented by a scheme (or algebraic stack) $\mathfrak{X}$ in the setting of E_∞-rings (see, for example, [32], [31], or [23]). A good understanding of the derived moduli problem X (or, equivalently, the geometric object $\mathfrak{X}$) is often helpful for analyzing $\mathfrak{X}_0$.

For example, let Y be a smooth algebraic variety over the complex numbers, and let $\overline{\mathcal{M}}_g(Y)$ denote the Kontsevich moduli stack of curves of genus g equipped with a stable map to Y (see, for example, [12]). Then $\overline{\mathcal{M}}_g(Y)$ represents a functor $X_0 : \mathrm{Ring} \to \mathrm{Gpd}$ which admits a natural enhancement $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$. This enhancement contains a great deal of useful information about the original moduli stack $\overline{\mathcal{M}}_g(Y)$: for example, it determines the *virtual fundamental class* of $\overline{\mathcal{M}}_g(Y)$ which plays an important role in Gromov-Witten theory.

(*b*) Let $G_{\mathbf{C}}$ be a reductive algebraic group over the complex numbers. Then $G_{\mathbf{C}}$ is canonically defined over the ring $\mathbf{Z}$ of integers. More precisely, there exists a split reductive group scheme $G_{\mathbf{Z}}$ over $\mathrm{Spec}\, \mathbf{Z}$ (well-defined up to isomorphism) such that $G_{\mathbf{Z}} \times \mathrm{Spec}\, \mathbf{C} \simeq G_{\mathbf{C}}$ ([4]). Since $\mathbf{Z}$ is the initial object in the category of commutative rings, the group scheme $G_{\mathbf{Z}}$ can be regarded as a "universal version" of the reductive algebraic group $G_{\mathbf{C}}$: it determines a reductive group scheme $G_R = G_{\mathbf{Z}} \times \mathrm{Spec}\, R$ over any commutative ring R. However, there are some suggestions that $G_{\mathbf{Z}}$ might admit an even more primordial description (for example, it has been suggested that we should regard the Weyl group W of $G_{\mathbf{C}}$ as the set of points $G(\mathbf{F}_1)$ of G with values in the "field with 1 element"; see [28]). The language of ring spectra provides one way of testing this hypothesis: the initial object in the ∞-category $\mathrm{CAlg}(\mathrm{Sp})$ is given by the sphere spectrum S, rather than the discrete ring $\mathbf{Z} \simeq \pi_0 S$. It therefore makes sense to ask: is the algebraic group $G_{\mathbf{C}}$ defined over the sphere spectrum? We will return to this question briefly in §10 (Remark 10.3).

(*c*) Let $X_0 : \mathrm{Ring} \to \mathrm{Gpd}$ be the classical moduli problem of Example 1.5, which assigns to each commutative ring R the groupoid $\mathrm{Hom}(\mathrm{Spec}\, R, \mathcal{M}_{1,1})$ of elliptic curves over R. It is possible to make sense of the notion of an elliptic curve over R when R is an arbitrary E_∞-ring, and thereby obtain an enhancement $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ of $\mathcal{M}_{\mathrm{Ell}}$. One can use this enhancement to give a moduli-theoretic reformulation of the Goerss-Hopkins-Miller theory of topological modular forms; we refer the reader to [21] for are more detailed discussion.

(*d*) The framework of derived moduli problems (or, more precisely, their formal analogues: see Definition 4.6) provides a good setting for the study of deformation theory. We will explain this point in more detail in the next section.

4. Formal Moduli Problems

Let $X : \mathrm{CAlg}_{\mathrm{Sp}} \to \mathcal{S}$ be a derived moduli problem. We define a *point* of X to be a pair $x = (k, \eta)$, where k is a field (regarded as a discrete E_∞-ring) and $\eta \in X(k)$. Our goal in this section is to study the local structure of the moduli problem X "near" the point x. More precisely, we will study the restriction of X to E_∞-rings which are closely related to the field k. To make this idea precise, we need to introduce a bit of terminology.

Definition 4.1. Let k be a field. We let CAlg_k denote the ∞-category whose objects are E_∞-rings A equipped with a map $k \to A$, where morphisms are given by commutative triangles

$$\begin{array}{ccc} & k & \\ \swarrow & & \searrow \\ A & \longrightarrow & A'. \end{array}$$

We will refer to the objects of CAlg_k as *E_∞-algebras over k*.

Remark 4.2. We say that a k-algebra A is *discrete* if it is discrete as an E_∞-ring: that is, if the homotopy groups $\pi_i A$ vanish for $i \neq 0$. The discrete k-algebras determine a full subcategory of CAlg_k, which is equivalent to the ordinary category of commutative rings A with a map $k \to A$.

Remark 4.3. Let k be a field. The category Chain_k of chain complexes over k admits a symmetric monoidal structure, given by the usual tensor product of chain complexes. A commutative algebra in the category Chain_k is called a *commutative differential graded algebra* over k. The functor $\mathrm{Chain}_k \to \mathrm{Mod}_k$ is symmetric monoidal, and determines a functor $\phi : \mathrm{CAlg}(\mathrm{Chain}_k) \to \mathrm{CAlg}(\mathrm{Mod}_k) \simeq \mathrm{CAlg}_k$. We say that a morphism $f : A_\bullet \to B_\bullet$ in $\mathrm{CAlg}_k^{\mathrm{dg}}$ is a *quasi-isomorphism* if it induces a quasi-isomorphism between the underlying chain complexes of $A_\bullet$ and $B_\bullet$. The functor ϕ carries every quasi-isomorphism of commutative differential graded algebras to an equivalence in CAlg_k. If k is a field of characteristic zero, then ϕ induces an equivalence $\mathrm{CAlg}(\mathrm{Chain}_k)[W^{-1}] \simeq \mathrm{CAlg}_k$, where W is the collection of quasi-isomorphisms: in other words, we can think of the ∞-category of E_∞-algebras over k as obtained from the ordinary category of commutative differential graded k-algebras by formally inverting the collection of quasi-isomorphisms.

Definition 4.4. Let k be a field and let $V \in \mathrm{Mod}_k$ be a k-module spectrum. We will say that V is *small* if the following conditions are satisfied:

(1) For every integer n, the homotopy group $\pi_n V$ is finite dimensional as a k-vector space.
(2) The homotopy groups $\pi_n V$ vanish for $n < 0$ and $n \gg 0$.

Let A be an E_∞-algebra over k. We will say that A is *small* if it is small as a k-module spectrum, and satisfies the following additional condition:

(3) The commutative ring $\pi_0 A$ has a unique maximal ideal $\mathfrak{p}$, and the map
$$k \to \pi_0 A \to \pi_0 A/\mathfrak{p}$$
is an isomorphism.

We let $\mathrm{Mod}_{\mathrm{sm}}$ denote the full subcategory of Mod_k spanned by the small k-module spectra, and $\mathrm{CAlg}_{\mathrm{sm}}$ denote the full subcategory of CAlg_k spanned by the small E_∞-algebras over k.

Remark 4.5. Let A be a small E_∞-algebra over k. Then there is a unique morphism $\epsilon : A \to k$ in CAlg_k; we will refer to ϵ as the *augmentation* on A.

Let $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ be a derived moduli problem, and let $x = (k, \eta)$ be a point of X. We define a functor $X_x : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ as follows: for every small E_∞-algebra A over k, we let $X_x(A)$ denote the fiber of the map $X(A) \to X(k)$ (induced by the augmentation $\epsilon : A \to k$) over the point η. The intuition is that X_x encodes the local structure of the derived moduli problem X near the point x.

Let us now axiomatize the expected behavior of the functor X_x:

Definition 4.6. Let k be a field. A *formal moduli problem over k* is a functor $X : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ with the following properties:

(1) The space $X(k)$ is contractible.
(2) Suppose that $\phi : A \to B$ and $\phi' : A' \to B$ are maps between small E_∞-algebras over k which induce surjections $\pi_0 A \to \pi_0 B$, $\pi_0 A' \to \pi_0 B$. Then the canonical map

$$X(A \times_B A') \to X(A) \times_{X(B)} X(A')$$

is a homotopy equivalence.

Remark 4.7. Let X be a derived moduli problem and let $x = (k, \eta)$ be a point of X. Then the functor $X_x : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ automatically satisfies condition (1) of Definition 4.6. Condition (2) is not automatic, but holds whenever the functor X is defined in a sufficiently "geometric" way. To see this, let us imagine that there exists some ∞-category of geometric objects $\mathcal{G}$ with the following properties:

(a) To every small k-algebra A we can assign an object $\mathrm{Spec}\, A \in \mathcal{G}$, which is contravariantly functorial in A.
(b) There exists an object $\mathcal{X} \in \mathcal{G}$ which represents X, in the sense that $X(A) \simeq \mathrm{Hom}_{\mathcal{G}}(\mathrm{Spec}\, A, \mathcal{X})$ for every small k-algebra A.

To verify that X_x satisfies condition (2) of Definition 4.6, it suffices to show that when $\phi : A \to B$ and $\phi : A' \to B$ are maps of small E_∞ algebras over k which induce surjections $\pi_0 A \to \pi_0 B \leftarrow \pi_0 A'$, then the diagram

$$\begin{array}{ccc} \mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, A' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, A & \longrightarrow & \mathrm{Spec}(A \times_B A') \end{array}$$

is a pushout square in $\mathcal{G}$. This assumption expresses the idea that $\mathrm{Spec}(A \times_B A')$ should be obtained by "gluing" $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ together along the common closed subobject $\mathrm{Spec}\, B$.

For examples of ∞-categories $\mathcal{G}$ satisfying the above requirements, we refer the reader to the work of Toën and Vezzosi on derived stacks (see, for example, [32]).

Remark 4.8. Let $X : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. Then X determines a functor $\overline{X} : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}\mathrm{et}$, given by the formula $\overline{X}(A) = \pi_0 X(A)$. It follows from condition (2) of Definition 4.6 that if we are given maps of small E_∞-algebras $A \to B \leftarrow A'$ which induce surjections $\pi_0 A \to \pi_0 B \leftarrow \pi_0 A'$, then the induced map

$$\overline{X}(A \times_B A') \to \overline{X}(A) \times_{\overline{X}(B)} \overline{X}(A')$$

is a surjection of sets (in fact, this holds under weaker assumptions: see Remark 6.19). There is a substantial literature on set-valued moduli functors of this type; see, for example, [24] and [18].

5. Tangent Complexes

Let $X : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ be a classical moduli problem. Let k be a field and let $\eta \in X(k)$, so that the pair $x = (k, \eta)$ can be regarded as a point of X. Following Grothendieck, we define the *tangent space* $T_{X,x}$ to be the fiber of the map $X(k[\epsilon]/(\epsilon^2)) \to X(k)$ over the point η. Under very mild assumptions, one can show that this fiber has the structure of a vector space over k: for example, if $\lambda \in k$ is a scalar, then the action of λ on $T_{X,x}$ is induced by the ring homomorphism $k[\epsilon]/(\epsilon^2) \to k[\epsilon]/(\epsilon^2)$ given by $\epsilon \mapsto \lambda\epsilon$.

Now suppose that $X : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem over a field k. Then $X(k[\epsilon]/(\epsilon^2)) \in \mathcal{S}$ is a topological space, which we will denote by $T_X(0)$. As in the classical case, $T_X(0)$ admits a great deal of algebraic structure. To see this, we need to introduce a bit of notation.

Let k be a field and let V be a k-module spectrum. We let $k \oplus V$ denote the direct sum of k and V (as a k-module spectrum). We will regard $k \oplus V$ as an E_∞-algebra over k, with a "square-zero" multiplication on the submodule V. Note that if V is a small k-module, then $k \oplus V$ is a small k-algebra (Definition 4.4). For each integer $n \geq 0$, we let $k[n]$ denote the n-fold shift of k as a k-module spectrum: it is characterized up to equivalence by the requirement

$$\pi_i k[n] \simeq \begin{cases} k & \text{if } i = n \\ 0 & \text{if} i \neq n \end{cases}$$

If X is a formal moduli problem over k, we set $T_X(n) = X(k \oplus k[n])$ (this agrees with our previous definition in the case $n = 0$). For $n > 0$, we have a pullback diagram of E_∞-algebras

$$\begin{array}{ccc} k \oplus k[n-1] & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

which, using conditions (1) and (2) of Definition 4.6, gives a pullback diagram

$$\begin{array}{ccc} T_X(n-1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & T_X(n) \end{array}$$

in the ∞-category of spaces. That is, we can identify $T_X(n-1)$ with the loop space of $T_X(n)$, so that the sequence $\{T_X(n)\}_{n \geq 0}$ can be regarded as a spectrum, which we will denote by T_X. We will refer to T_X as the *tangent complex* to the formal moduli problem X.

In fact, we can say more: the spectrum T_X admits the structure of a module over k. Roughly speaking, this module structure comes from the following construction: for each scalar $\lambda \in k$, multiplication by λ induces a map from $k[n]$ to itself, and therefore a map from $T_X(n)$ to itself; these maps are compatible with one another and give an action of k on the spectrum T_X.

Remark 5.1. Here is a more rigorous construction of the k-module structure on the tangent complex T_X. We say that a functor $U : \mathrm{Mod}_{\mathrm{sm}} \to \mathcal{S}$ is *excisive* if it satisfies the following linear version of the conditions of Definition 4.6:

(1) The space $U(0)$ is contractible.
(2) For every pushout diagram

$$\begin{array}{ccc} V & \longrightarrow & V' \\ \downarrow & & \downarrow \\ W & \longrightarrow & W' \end{array}$$

in the ∞-category $\mathrm{Mod}_{\mathrm{sm}}$, the induced diagram of spaces

$$\begin{array}{ccc} U(V) & \longrightarrow & U(V') \\ \downarrow & & \downarrow \\ U(W) & \longrightarrow & U(W') \end{array}$$

is a pullback square.

If $W \in \mathrm{Mod}_k$ is an arbitrary k-module spectrum, then the construction $V \mapsto \mathrm{Hom}_k(V^\vee, W)$ gives an excisive functor from $\mathrm{Mod}_{\mathrm{sm}}$ to $\mathcal{S}$ (here $V^\vee$ denotes the k-linear dual of V). In fact, every excisive functor arises in this way: the above construction determines a fully faithful embedding $\mathrm{Mod}_k \hookrightarrow \mathrm{Fun}(\mathrm{Mod}_{\mathrm{sm}}, \mathcal{S})$ whose essential image is the collection of excisive functors.

If $X : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ is a formal moduli problem, then one can show that the functor $V \mapsto X(k \oplus V)$ is excisive. It follows that there exists a k-module spectrum W (which is determined uniquely up to equivalence) for which $X(k \oplus V) \simeq \mathrm{Hom}_k(V^\vee, W)$. This k-module spectrum W can be identified with the tangent complex T_X; for example, we have

$$\Omega^\infty T_X = T_X(0) = X(k \oplus k[0]) \simeq \mathrm{Hom}_k(k[0]^\vee, W) \simeq \Omega^\infty W$$

Remark 5.2. The tangent complex to a formal moduli problem X carries a great deal of information about X. For example, if $\alpha : X \to X'$ is a natural transformation between formal moduli problems, then α is an equivalence if and only if it induces a homotopy equivalence of k-module spectra $T_X \to T_{X'}$. In concrete terms, this means that if α induces a homotopy equivalence $X(k \oplus k[n]) \to X'(k \oplus k[n])$ every integer $n \geq 0$, then α induces a homotopy equivalence $F(A) \to F'(A)$ for every small E_∞-algebra A over k. This follows from the fact that A admits a "composition series"

$$A = A(m) \to A(m-1) \to \cdots \to A(0) = k,$$

where each of the maps $A(j) \to A(j-1)$ fits into a pullback diagram

$$\begin{array}{ccc} A(j) & \longrightarrow & A(j-1) \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n_j] \end{array}$$

for some $n_j > 0$.

Remark 5.2 suggests that it should be possible to reconstruct a formal moduli problem X from its tangent complex T_X. If k is a field of characteristic zero, then mathematical folklore asserts that every formal moduli problem is "controlled" by a differential graded Lie algebra over k. This can be formulated more precisely as follows:

Theorem 5.3. *Let k be a field of characteristic zero, and let* Moduli *denote the full subcategory of* $\mathrm{Fun}(\mathrm{CAlg}_{\mathrm{sm}}, \mathcal{S})$ *spanned by the formal moduli problems over k. Then there is an equivalence of ∞-categories* $\Phi : \mathrm{Moduli} \to \mathrm{Lie}_k^{\mathrm{dg}}$*, where* $\mathrm{Lie}_k^{\mathrm{dg}}$ *denotes the ∞-category of differential graded Lie algebras over k (Example 2.19). Moreover, if* $U : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Mod}_k$ *denotes the forgetful functor (which assigns to each differential graded Lie algebra its underlying chain complex), then the composition $U \circ \Phi$ can be identified with the functor $X \mapsto T_X[-1]$.*

In other words, if X is a formal moduli problem, then the shifted tangent complex $T_X[-1] \in \mathrm{Mod}_k$ can be realized as a differential graded Lie algebra over k. Conversely, every differential graded Lie algebra over k arises in this way (up to quasi-isomorphism).

Remark 5.4. The functor $\Phi^{-1} : \mathrm{Lie}_k^{\mathrm{dg}} \to \mathrm{Moduli} \subseteq \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{sm}}, \mathcal{S})$ is constructed by Hinich in [14]. Roughly speaking, if $\mathfrak{g}$ is a differential graded Lie algebra and A is a small E_∞-algebra over k, then $\Phi^{-1}(\mathfrak{g})(A)$ is the space of solutions to the Maurer-Cartan equation $dx = [x, x]$ in the differential graded Lie algebra $\mathfrak{g} \otimes_k \mathfrak{m}_A$.

Remark 5.5. The notion that differential graded Lie algebras should play an important role in the description of moduli spaces goes back to Quillen's work on rational homotopy theory ([33]), and was developed further in unpublished work of Deligne, Drinfeld, and Feigin. Many mathematicians have subsequently taken up these ideas: see, for example, the book of Kontsevich and Soibelman ([18]).

Remark 5.6. For applications of Theorem 5.3 to the classification of deformations of algebraic structures, we refer the reader to [15] and [17].

Remark 5.7. In §8, we will sketch the proof of a "noncommutative" version of Theorem 5.3 (Theorem 6.20). Theorem 5.3 can be proven using the same strategy; see Remark 8.22.

Remark 5.8. Suppose that R is a commutative k-algebra equipped with an augmentation $\epsilon : R \to k$. Then R defines a formal moduli problem X over k, which carries a small E_∞-algebra A over k to the fiber of the map

$$\mathrm{Hom}_{\mathrm{CAlg}_k}(R, A) \to \mathrm{Hom}_{\mathrm{CAlg}_k}(R, k).$$

When k is of characteristic zero, the tangent complex T_X can be identified with the complex Andre-Quillen cochains taking values in k. In this case, the existence of a natural differential graded Lie algebra structure on $T_X[-1]$ is proven in [26].

Remark 5.9. Here is a heuristic explanation of Theorem 5.3. Let $X : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. Since every k-algebra A comes equipped with a canonical map $k \to A$, we get an induced map $* \simeq X(k) \to X(A)$: in other words, each of the spaces $X(A)$ comes equipped with a natural base point. We can then define a new functor $\Omega X : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ by the formula $(\Omega X)(A) = \Omega X(A)$ (here Ω denotes the loop space functor from the ∞-category of pointed spaces to itself). Then ΩX is another formal moduli problem, and an elementary calculation gives $T_{\Omega X} \simeq T_X[-1]$. However, ΩX is equipped with additional structure: composition of loops gives a multiplication on ΩX (which is associative up to coherent homotopy), so we can think of ΩX as a *group object* in the ∞-category of formal moduli problems.

In classical algebraic geometry, the tangent space to an algebraic group G at the origin admits a Lie algebra structure. In characteristic zero, this Lie algebra structure permits us to reconstruct the formal completion of G (via the Campbell-Hausdorff formula). Theorem 5.3 can be regarded as an analogous statement in the context of formal moduli problems: the group structure on ΩX determines a Lie algebra structure on its tangent complex $T_{\Omega X} \simeq T_X[-1]$. Since we are working in a formal neighborhood of a fixed point, allows us to reconstruct the group ΩX (and, with a bit more effort, the original formal moduli problem X).

Example 5.10. Let $X : \mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ be the formal moduli problem of Example 3.7, which assigns to every E_∞-ring A the ∞-groupoid $F(A)$ of projective A-modules of rank n. Giving a point $x = (k, \eta)$ of X is equivalent to giving a field k together with a vector space V_0 of dimension n over k. In this case, the functor $X_x : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ can be described as follows: to every small k-algebra A, the functor X_x assigns the ∞-category of pairs (V, α), where V is a projective A-module of rank n and $\alpha : k \wedge_A V \to V_0$ is an isomorphism of k-vector spaces. It is not difficult to show that X_x is a formal moduli problem in the sense of Definition 4.6. We will denote its tangent complex $T_{X,x}$.

Unwinding the definitions, we see that $T_{X,x}(0) = X_x(k[\epsilon]/(\epsilon^2))$ can be identified with a classifying space for the groupoid of projective $k[\epsilon]/(\epsilon^2)$-modules V which deform V_0. This groupoid has only one object up to isomorphism, given by the tensor product $k[\epsilon]/(\epsilon^2) \otimes_k V_0$. It follows that $T_{X,x}(0)$ can be identified with the classifying space BG for the group G of automorphisms of $k[\epsilon]/(\epsilon^2) \otimes_k V_0$ which reduce to the identity moduli ϵ. Such an automorphism can be written as $1 + \epsilon M$, where $M \in \mathrm{End}(V_0)$. Consequently, $T_{X,x}(0)$ is homotopy equivalent to the classifying space for the k-vector space $\mathrm{End}_k(V_0)$, regarded as a group under addition.

Amplifying this argument, we obtain an equivalence of k-module spectra $T_{X,x} \simeq \mathrm{End}_k(V_0)[1]$. The shifted tangent complex $T_{X,x}[-1] \simeq \mathrm{End}_k(V_0)$ has the structure of a Lie algebra over k (and therefore of a differential graded Lie algebra over k, with trivial grading and differential), given by the usual commutator bracket of endomorphisms.

6. Noncommutative Geometry

Our goal in this paper is to describe an analogue of Theorem 5.3 in the setting of noncommutative geometry. We begin by describing a noncommutative analogue of the theory of E_∞-rings.

Definition 6.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We can associate to $\mathcal{C}$ a new ∞-category $\mathrm{Alg}(\mathcal{C})$ of *associative algebra* objects of $\mathcal{C}$. The ∞-category $\mathrm{Alg}(\mathcal{C})$ inherits the structure of a symmetric monoidal ∞-category. We can therefore define a sequence of ∞-categories $\mathrm{Alg}^{(n)}(\mathcal{C})$ by induction on n:

- (a) If $n = 1$, we let $\mathrm{Alg}^{(n)}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C})$.
- (b) If $n > 1$, we let $\mathrm{Alg}^{(n)}(\mathcal{C}) = \mathrm{Alg}(\mathrm{Alg}^{(n-1)}(\mathcal{C}))$.

We will refer to $\mathrm{Alg}^{(n)}(\mathcal{C})$ as the *∞-category of E_n-algebras in $\mathcal{C}$*.

Remark 6.2. We can summarize Definition 6.1 informally as follows: an E_n-algebra object of a symmetric monoidal ∞-category $\mathcal{C}$ is an object $A \in \mathcal{C}$ which is equipped with n multiplication operations $\{m_i : A \otimes A \to A\}_{1 \leq i \leq n}$; these multiplications are required to be associative and unital (up to coherent homotopy) and to be compatible with one another in a suitable sense.

Example 6.3. Let $\mathcal{C} = \mathcal{S}$ be the ∞-category of spaces, endowed with the symmetric monoidal structure given by Cartesian products of spaces. For every pointed space X, the loop space ΩX has the structure of an algebra object of $\mathcal{S}$: the multiplication on ΩX is given by concatenation of loops. In fact, we can say a bit more: the algebra object $\Omega X \in \mathrm{Alg}(\mathcal{S})$ is *grouplike*, in the sense that the multiplication on $\Omega(X)$ determines a group structure on the set $\pi_0 \Omega(X) \simeq \pi_1 X$. This construction determines an equivalence from the ∞-category of *connected* pointed spaces to the full subcategory of $\mathrm{Alg}(\mathcal{C})$ spanned by the *grouplike* associative algebras.

More generally, the construction $X \mapsto \Omega^n X$ establishes an equivalence between the ∞-category of $(n-1)$-connected pointed spaces and the full subcategory of grouplike E_n-algebras of $\mathcal{S}$. See [25] for further details.

Example 6.4. Fix an E_∞-ring k, and let $\mathrm{Mod}_k = \mathrm{Mod}_k(\mathrm{Sp})$ denote the ∞-category of k-module spectra. Then Mod_k admits a symmetric monoidal structure, given by the relative smash product $(M, N) \mapsto M \wedge_k N$. We will refer to E_n-algebra objects of Mod_k as *E_n-algebras over k*. We let $\mathrm{Alg}_k^{(n)} = \mathrm{Alg}^{(n)}(\mathrm{Mod}_k)$ denote the ∞-category of E_n-algebras over k. When k is the sphere spectrum S, we will refer to an E_n-algebra over k simply as an *E_n-ring*.

Remark 6.5. For any symmetric monoidal ∞-category $\mathcal{C}$, there is a forgetful functor $\mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$, which assigns to an associative algebra its underlying object of $\mathcal{C}$. These forgetful functors determine rise to a tower of ∞-categories

$$\cdots \to \mathrm{Alg}^{(3)}(\mathcal{C}) \to \mathrm{Alg}^{(2)}(\mathcal{C}) \to \mathrm{Alg}^{(1)}(\mathcal{C}).$$

The inverse limit of this tower can be identified with the ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects of $\mathcal{C}$.

Remark 6.6. There is a non-inductive description of the ∞-category $\mathrm{Alg}^{(n)}(\mathcal{C})$ of E_n-algebra objects in $\mathcal{C}$: it can be obtained as the ∞-category of representations in $\mathcal{C}$ of the *little n-cubes* operad introduced by Boardman and Vogt; see [2].

Remark 6.7. It is convenient to extend Definition 6.1 to the case $n = 0$: an E_0-algebra object of $\mathcal{C}$ is an object $A \in \mathcal{C}$ which is equipped with a distinguished map $\mathbf{1} \to A$, where $\mathbf{1}$ denotes the unit with respect to the tensor product on $\mathcal{C}$.

Remark 6.8. When $\mathcal{C}$ is an ordinary category, Definition 6.1 is somewhat degenerate: the categories $\mathrm{Alg}^{(n)}(\mathcal{C})$ coincide with $\mathrm{CAlg}(\mathcal{C})$ for $n \geq 2$. This is a consequence of the classical Eckmann-Hilton argument: if $A \in \mathcal{C}$ is equipped with two commuting unital multiplication operations m_1 and m_2, then m_1 and m_2 are commutative and coincide with one another. If $\mathcal{C}$ is the category of sets, the proof can be given as follows. Since the unit map $\mathbf{1} \to A$ for the multipication m_1 is a homomorphism with multiplication m_2, we see that the unit elements of A for the multiplications m_1 and m_2 coincide with a single element $u \in A$. Then

$$m_1(a, b) = m_1(m_2(a, u), m_2(u, b)) = m_2(m_1(a, u), m_1(u, b)) = m_2(a, b).$$

A similar calculation gives $m_1(a, b) = m_2(b, a)$, so that $m_1 = m_2$ is commutative.

Remark 6.9. Let k be a field, and let Chain_k be the ordinary category of chain complexes over k. The functor $\mathrm{Chain}_k \to \mathrm{Mod}_k$ of Remark 3.2 is symmetric monoidal: in other words, the relative smash product $\wedge_k$ is compatible with the usual tensor product of chain complexes. In particular, we get a functor $\theta : \mathrm{Alg}(\mathrm{Chain}_k) \to \mathrm{Alg}(\mathrm{Mod}_k) = \mathrm{Alg}_k^{(1)}$. The category $\mathrm{Alg}(\mathrm{Chain}_k)$ can be identified with the category of *differential graded algebras over k*. We say that a map of differential graded algebras $f : A_\bullet \to B_\bullet$ is a quasi-isomorphism if it induces a quasi-isomorphism between the underlying chain complexes of $A_\bullet$ and $B_\bullet$; in this case, the morphism $\theta(f)$ is an equivalence in $\mathrm{Alg}_k^{(1)}$. Let W be the collection of quasi-isomorphisms between differential graded algebras. One can show that θ induces an equivalence $\mathrm{Alg}(\mathrm{Chain}_k)[W^{-1}] \to \mathrm{Alg}_k^{(1)}$: that is, E_1-algebras over a field k (of any characteristic) can be identified with differential graded algebras over k.

Remark 6.10. Let k be a field and let A be an E_n-algebra over k. If $n \geq 1$, then A has an underlying associative multiplication. This multiplication endows $\pi_* A$ with the structure of a graded algebra over k. In particular, $\pi_0 A$ is an associative k-algebra.

Definition 6.11. Let k be a field and let A be an E_n-algebra over k, where $n \geq 1$. We will say that A is *small* if the following conditions are satisfied:

(1) The algebra A is small when regarded as a k-module spectrum: that is, the homotopy groups $\pi_i A$ are finite dimensional, and vanish if $i < 0$ or $i \gg 0$.
(2) Let $\mathfrak{p}$ be the radical of the (finite-dimensional) associative k-algebra $\pi_0 A$. Then the composite map $k \to \pi_0 A \to \pi_0 A/\mathfrak{p}$ is an isomorphism.

We let $\mathrm{Alg}_{\mathrm{sm}}^{(n)}$ denote the full subcategory of $\mathrm{Alg}_k^{(n)}$ spanned by the small E_n-algebras over k.

Remark 6.12. Let A be an E_n-algebra over a field k. An *augmentation* on A is a map of E_n-algebras $A \to k$. The collection of augmented E_n-algebras over k can be organized into an ∞-category, which we will denote by $\mathrm{Alg}_{\mathrm{aug}}^{(n)}$. If $n \geq 1$ and A is a small E_n-algebra over k, then A admits a unique augmentation $A \to k$ (up to a contractible space of choices). Consequently, we can view $\mathrm{Alg}_{\mathrm{sm}}^{(n)}$ as a full subcategory of $\mathrm{Alg}_{\mathrm{aug}}^{(n)}$.

If $\eta : A \to k$ is an augmented E_n-algebra over k, we let $\mathfrak{m}_A$ denote the fiber of the map η. We will refer to $\mathfrak{m}_A$ as the *augmentation ideal* of A.

Remark 6.13. If $n = 0$, then an augmentation on an E_0-algebra $A \in \mathrm{Alg}_k^{(0)}$ is a map of k-module spectra $\eta : A \to k$ which is left inverse to the unit map $k \to A$. The construction $(\eta : A \to k) \mapsto \mathfrak{m}_A$ determines an equivalence of ∞-categories $\mathrm{Alg}_{\mathrm{aug}}^{(0)} \simeq \mathrm{Mod}_k$.

It is convenient to extend Definition 6.11 to the case $n = 0$. We say that an augmented E_0-algebra A is *small* if A (or, equivalently, the augmentation ideal $\mathfrak{m}_A$) is small when regarded as a k-module spectrum. We let $\mathrm{Alg}_{\mathrm{sm}}^{(0)} \subseteq \mathrm{Alg}_{\mathrm{aug}}^{(0)} \simeq \mathrm{Mod}_k$ denote the full subcategory spanned by the small E_0-algebras over k.

The following elementary observation will be used several times in this paper:

Claim 6.14. *Let $f : A \to B$ be a map of small E_n-algebras over k which induces a surjection $\pi_0 A \to \pi_0 B$. Then there exists a sequence of maps*

$$A = A(0) \to A(1) \to \cdots \to A(m) = B$$

with the following property: for each integer $0 \leq i < m$, there is a pullback diagram of small E_n-algebras

$$\begin{array}{ccc} A(i) & \longrightarrow & A(i+1) \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[j] \end{array}$$

for some $j > 0$ (in other words, $A(i)$ can be identified with the fiber of some map $A(i+1) \to k \oplus k[j]$).

Remark 6.15. Claim 6.14 is most useful in the case where f is the augmentation map $A \to k$. We will refer to a sequence of maps

$$A = A(0) \to A(1) \to \cdots \to A(m) \simeq k$$

satisfying the requirements of Claim 6.14 as a *composition series* for A.

Definition 6.16. Let k be a field and let $n \geq 0$ be an integer. A *formal E_n moduli problem* over k is a functor $X : \mathrm{Alg}_{\mathrm{sm}}^{(n)} \to \mathcal{S}$ with the following properties:

(1) The space $X(k)$ is contractible.

(2) Suppose we are given a pullback diagram of small E_n-algebras

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

such that the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective. Then the diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B) \end{array}$$

is a pullback diagram in $\mathcal{S}$.

Remark 6.17. Every formal E_n moduli problem $X : \mathrm{Alg}^{(n)}_{\mathrm{sm}} \to \mathcal{S}$ determines a formal moduli problem X' in the sense of Definition 4.6, where X' is given by the composition

$$\mathrm{Alg}_{\mathrm{sm}} \to \mathrm{Alg}^{(n)}_{\mathrm{sm}} \xrightarrow{X} \mathcal{S}\,.$$

We define the *tangent complex of* X to be the tangent complex of X', as defined in §5. We will denote the tangent complex of X by $T_X \in \mathrm{Mod}_k$.

Remark 6.18. Let X be as in Definition 6.16. By virtue of Claim 6.14, it suffices to check condition (2) in the special case where $A = k$ and $B = k \oplus k[j]$, for some $j > 0$. In other words, condition (2) is equivalent to the requirement that for every map $B' \to k \oplus k[j]$, we have a fiber sequence

$$X(B \times_{k \oplus k[j]} k) \to X(B) \to X(k \oplus k[j]).$$

The final term in this sequence can be identified with $T_X(j) = \Omega^\infty(T_X[j])$.

Remark 6.19. The argument of Remark 6.18 shows that condition (2) of Definition 6.16 is equivalent to the following apparently stronger condition:

(2′) Suppose we are given a pullback diagram of small E_n-algebras

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

such that the maps $\pi_0 A \to \pi_0 B$ is surjective. Then the diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B) \end{array}$$

is a pullback diagram in $\mathcal{S}$.

Let V_0 be a finite dimensional vector space over k, and let $X_x : \mathrm{CAlg}_{\mathrm{sm}} \to \mathcal{S}$ be the formal moduli problem of Example 5.10, so that X_x assigns to every small E_∞-algebra A over k the ∞-groupoid of pairs (V, α), where V is an A-module and $\alpha : k \wedge_A V \simeq V_0$ is an equivalence. The definition of X_x does not make any use of the commutativity of A. Consequently, X_x extends naturally to a functor $\widehat{X}_x : \mathrm{Alg}^{(1),}_{\mathrm{sm}} \to \mathcal{S}$, By definition, the shifted tangent complex of $\widehat{X}_x[-1]$ is given by the Lie algebra $T_{X_x}[-1] \simeq \mathrm{End}(V_0)$. If k is of characteristic zero, then Theorem 5.3 implies that the formal moduli problem X_x can be canonically reconstructed from the vector space $\mathrm{End}(V_0)$ together with its Lie algebra structure. However, the formal E_1 moduli problem $\widehat{X}_x$ is additional data, since we can evaluate $\widehat{X}_x$ on algebras which are not necessarily commutative. Consequently, it is natural to expect the existence of $\widehat{X}_x$ to be reflected in some additional structure on the Lie algebra $\mathrm{End}(V_0)$. We observe that $\mathrm{End}(V_0)$ is not merely a Lie algebra: there is an associative product (given by composition) whose commutator gives the Lie bracket on $\mathrm{End}(V_0)$. In fact, this is a general phenomenon:

Theorem 6.20. *Let k be a field, let $n \geq 0$, and let Moduli_n be the full subcategory of $\mathrm{Fun}(\mathrm{Alg}^{(n)}_{\mathrm{sm}}, \mathcal{S})$ spanned by the formal E_n moduli problems. Then there exists an equivalence of ∞-categories $\Phi : \mathrm{Moduli}_n \to \mathrm{Alg}^{(n)}_{\mathrm{aug}}$. Moreover, if $U : \mathrm{Alg}^{(n)}_{\mathrm{aug}} \to \mathrm{Mod}_k$ denotes the forgetful functor $A \mapsto \mathfrak{m}_A$ which assigns to each augmented E_n-algebra its augmentation ideal, then the composition $U \circ \Phi$ can be identified with the functor $X \mapsto T_X[-n]$.*

In other words, if X is a formal E_n-module problem, then the shifted tangent complex $T_X[-n]$ can be identified with the augmentation ideal in an augmented E_n-algebra A: that is, $T_X[-n]$ admits a nonunital E_n-algebra structure. Moreover, this structure determines the formal E_n moduli problem up to equivalence.

Example 6.21. Suppose that $n = 0$. The construction $V \mapsto k \oplus V$ determines an equivalence $\mathrm{Mod}_{\mathrm{sm}} \simeq \mathrm{Alg}^{(0)}_{\mathrm{sm}}$. Under this equivalence, we can identify the ∞-category Moduli_0 of formal E_0 moduli problems with the full subcategory of $\mathrm{Fun}(\mathrm{Mod}_{\mathrm{sm}}, \mathcal{S})$ spanned by the excisive functors (see Remark 5.1). In this case, Theorem 6.20 reduces to the claim of Remark 5.1: every excisive functor $U : \mathrm{Mod}_{\mathrm{sm}} \to \mathcal{S}$ has the form $V \mapsto \mathrm{Hom}_k(V^\vee, W) \simeq \Omega^\infty(V \wedge_k W)$ for some object $W \in \mathrm{Mod}_k$, which is determined up to equivalence. Note that we can identify W with the tangent complex to the formal E_0 moduli problem $A \mapsto U(\mathfrak{m}_A)$.

Remark 6.22. Unlike Theorem 5.3, Theorem 6.20 does not require any assumption on the characteristic of the ground field k.

We conclude this section by observing that Remark 5.2 holds in the noncommutative context:

Proposition 6.23. *Let $\alpha : X \to X'$ be a map of formal E_n moduli problems, and suppose that the induced map $T_X \to T_{X'}$ is an equivalence of k-module spectra. Then α is an equivalence.*

Proof. Let A be a small E_n-algebra over k; we wish to prove that α induces a homotopy equivalence $X(A) \to X'(A)$. Using Claim 6.14, we can choose a composition series

$$A = A(0) \to A(1) \to \cdots \to A(m) = k$$

for A. We will prove that α induces a homotopy equivalence $\theta_i : X(A(i)) \to X'(A(i))$ using descending induction on i. The case $i = m$ is trivial. Assume that $0 \leq i < m$ and that θ_{i+1} is a homotopy equivalence; we will prove that θ_i is a homotopy equivalence. By definition, we have a fiber sequence of small E_m-algebras

$$A(i) \to A(i+1) \to k \oplus k[j]$$

for some $j > 0$. This gives rise to a map of fiber sequences

$$\begin{array}{ccccc} X(A(i)) & \longrightarrow & X(A(i+1)) & \longrightarrow & T_X(j) \\ \downarrow \theta_i & & \downarrow \theta_{i+1} & & \downarrow \theta' \\ X'(A(i)) & \longrightarrow & X'(A(i+1)) & \longrightarrow & T_{X'}(j). \end{array}$$

The inductive hypothesis implies that θ_{i+1} is a homotopy equivalence, and our assumption implies that θ' is a homotopy equivalence; it follows that θ_i is a homotopy equivalence as well. ☐

7. Koszul Duality

Fix a field k and an integer $n \geq 0$. Theorem 6.20 asserts the existence of an equivalence of ∞-categories

$$\mathrm{Alg}^{(n)}_{\mathrm{aug}} \overset{\Phi^{-1}}{\simeq} \mathrm{Moduli}_n \subseteq \mathrm{Fun}(\mathrm{Alg}^{(n)}_{\mathrm{sm}}, \mathcal{S}).$$

The appearance of the theory of E_n-algebras on both sides of this equivalence is somewhat striking: it is a reflection of the Koszul self-duality of the little n-cubes operad (see [11]). In this section, we give a quick overview of Koszul duality, collecting the ideas which are needed to prove Theorem 6.20.

Definition 7.1. Let A be an E_n-algebra over a field k. We let $\mathrm{Aug}(A) = \mathrm{Hom}_{\mathrm{Alg}_k^{(n)}}(A, k) \in \mathcal{S}$ denote the space of augmentations on A. Suppose that A and B are E_n-algebras equipped with augmentations $\epsilon : A \to k$ and $\epsilon' : B \to k$. We let $\mathrm{Pair}(A, B) \in \mathcal{S}$ denote the homotopy fiber of the map of spaces $\mathrm{Aug}(A \wedge_k B, k) \to \mathrm{Aug}(A, k) \times \mathrm{Aug}(B, k)$. More informally: $\mathrm{Pair}(A, B)$ is the space of augmentations $\phi : A \wedge_k B \to k$ which are compatible with ϵ and ϵ'.

Example 7.2. Suppose that $n = 0$. Then the construction $V \mapsto k \oplus V$ defines an equivalence from the ∞-category Mod_k of k-module spectra to the ∞-category $\mathrm{Alg}^{(n)}_{\mathrm{aug}}$. If V and W are k-module spectra, then a pairing of $k \oplus V$ with $k \oplus W$ is a k-linear map

$$\phi : (k \oplus V) \wedge_k (k \oplus W) \simeq k \oplus V \oplus W \oplus (V \wedge_k W) \to k$$

such that $\phi|k = \mathrm{id}$ and $\phi|V = \phi|W = 0$. In other words, we can think of a pairing between $k \oplus V$ and $k \oplus W$ as a k-linear map $V \wedge_k W \to k$.

Example 7.3. Suppose that $n = 1$. If A is an E_1-algebra over k, then we can think of an augmentation on A as a map of associative k-algebras $A \to k \simeq \mathrm{End}_k(k)$: that is, as a left action of A on k. If we are given a pair of augmented E_1-algebras $\epsilon : A \to k$, $\epsilon' : B \to k$, then k can be regarded as a left A-module (via ϵ) and a right B^{op}-module (via ϵ'). To give a pairing of A with B is equivalent to promoting k to a left $A \wedge_k B$-module: in other words, it is the data which allows us to commute the left A-action on k with the right B^{op}-action on k, and thereby identify k with an A-B^{op} bimodule.

Claim 7.4. *Let A be an augmented E_n-algebra over a field k. Then the construction*

$$(B \in \mathrm{Alg}^{(n)}_{\mathrm{aug}}) \mapsto (\mathrm{Pair}(A, B) \in \mathcal{S})$$

is a representable functor. In other words, there exists an augmented E_n-algebra $\mathbb{D}(A)$ and a pairing $\phi \in \mathrm{Pair}(A, \mathbb{D}(A))$ with the following universal property: for every augmented E_n-algebra B over k, composition with ϕ induces a homotopy equivalence

$$\mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(B, \mathbb{D}(A)) \simeq \mathrm{Pair}(A, B).$$

We will refer to $\mathbb{D}(A)$ as the Koszul dual to A.

Remark 7.5. By the adjoint functor theorem, Claim 7.4 is equivalent to the assertion that the functor $B \mapsto \mathrm{Pair}(A, B)$ carries colimits in $\mathrm{Alg}^{(n)}_{\mathrm{aug}}$ to limits of spaces.

Remark 7.6. The construction $A, B \mapsto \mathrm{Pair}(A, B)$ is symmetric in A and B. Consequently, for any pair of augmented E_n-algebras A and B, we have homotopy equivalences

$$\mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(A, \mathbb{D}(B)) \simeq \mathrm{Pair}(A, B) \simeq \mathrm{Pair}(B, A) \simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(B, \mathbb{D}(A)).$$

Example 7.7. Let $V \in \mathrm{Mod}_k$ be a k-module spectrum, so that we can view $k \oplus V$ as an augmented E_0-algebra over k. It follows from Example 7.2 that the Koszul dual $\mathbb{D}(k \oplus V)$ can be identified with the augmented E_0-algebra $k \oplus V^\vee$, where $V^\vee$ denotes the k-linear dual of V; the homotopy groups of $V^\vee$ are given by $\pi_i V^\vee \simeq \mathrm{Hom}_k(\pi_{-i} V, k)$.

Example 7.8. Let A be an augmented E_1-algebra over k, so that we can view k as a left A-module. It follows from Example 7.3 that the Koszul dual $\mathbb{D}(A)$ can be identified with the E_1-algebra $\mathrm{End}_A(k)$ of A-linear endomorphisms of k: note that $\mathrm{End}_A(k)$ is universal among E_1-algebras with a left action on k commuting with our given left action of A on k.

Remark 7.9. Let A be an augmented E_n-algebra over k. The canonical pairing $\phi : A \otimes \mathbb{D}(A) \to k$ is classified by a map $f : A \to \mathbb{D}^2(A)$. We will refer to f as the *double duality* map of A. When $n = 0$, Example 7.7 implies that f is an equivalence if and only if each of the vector spaces $\pi_i A$ is finite-dimensional over k. More generally, it is natural to expect f to be an equivalence when the augmented E_n-algebra A satisfies some finiteness conditions, such as the smallness condition of Definition 6.11 (though weaker conditions will also suffice).

Notation 7.10. Fix a field k and an integer $n \geq 0$. We let $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Alg}_k^{(n)}$ be a left adjoint to the forgetful functor: that is, Free assigns to each k-module spectrum V the free E_n-algebra $\mathrm{Free}(V)$ on V. Note that zero map $V \to k$ determines an augmentation on $\mathrm{Free}(V)$: we will view $\mathrm{Free}(V)$ as an *augmented* E_n-algebra.

To prove Theorem 6.20, we will need the following facts about the Koszul duality functor $\mathbb{D}$ (which we assert here without proof):

$(K1)$ Let V and W be k-module spectra. Then every map $V \wedge_k W \to k$ induces a pairing of augmented E_n-algebras
$$(k \oplus V) \wedge_k \mathrm{Free}(W[-n]) \to k.$$

$(K2)$ Let $V = k[m]$, $W = k[-m]$, and let $\alpha : V \wedge_k W \simeq k$ be the canonical equivalence. If $m \geq 0$, then the induced pairing
$$(k \oplus V) \wedge_k \mathrm{Free}(W[-n]) \to k$$
is perfect: it induces equivalences
$$k \oplus k[m] \simeq \mathbb{D}\,\mathrm{Free}(k[-m-n]) \qquad \mathrm{Free}(k[-m-n]) \simeq \mathbb{D}(k \oplus k[m]).$$

$(K3)$ Suppose we are given a pullback diagram
$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$
of small E_n-algebras over k, where the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective. Then the induced diagram of E_n-algebras
$$\begin{array}{ccc} \mathbb{D}A' & \longleftarrow & \mathbb{D}A \\ \uparrow & & \uparrow \\ \mathbb{D}B' & \longleftarrow & \mathbb{D}B \end{array}$$
is a pushout square.

Remark 7.11. As in Remark 6.18, the general case of $(K3)$ can be deduced from the special case where $A = k$ and $B = k \oplus k[m]$ for some $m > 0$.

Remark 7.12. For every k-module spectrum W, the pairing of $(K1)$ gives an identification of the Koszul dual $\mathbb{D}\,\mathrm{Free}(W[-n])$ with $k \oplus W^\vee$. This can be deduced from $(K2)$ by resolving W as a colimit of k-module spectra of the form $k[-m]$ where $m \geq 0$. However, the adjoint map
$$\mathrm{Free}(W[-n]) \to \mathbb{D}^2\,\mathrm{Free}(W[-n]) \simeq \mathbb{D}(k \oplus W^\vee)$$
is generally not an equivalence without some additional restrictions on W.

Remark 7.13. Let A be an augmented E_1-algebra. According to Example 7.8, the Koszul dual of A can be identified with
$$\mathrm{End}_A(k) = \mathrm{Hom}_A(k, k) = \mathrm{Hom}_k(k \wedge_A k, k).$$
That is, $\mathbb{D}(A)$ can be identified with the k-linear dual of the *bar construction* $BA = k \wedge_A k$. The algebra structure on $\mathbb{D}(A)$ is determined by an associative coalgebra structure on $k \wedge_A k$, given by
$$BA = (k \wedge_A k) \simeq k \wedge_A A \wedge_A k \to k \wedge_A k \wedge_A k \simeq BA \wedge_k BA.$$

The construction $A \mapsto BA$ is a symmetric monoidal functor from augmented algebras in Mod_k to augmented coalgebras in Mod_k. Suppose now that A is an augmented E_n-algebra over k. Then we can view A as an E_{n-1}-algebra in the ∞-category of augmented algebras over k, so that BA is an E_{n-1}-algebra in the ∞-category of augmented coalgebras over k. If $n > 1$, then we can use the residual algebra structure on BA to perform the bar construction again. One can show that the Koszul dual $\mathbb{D}(A)$ is the k-linear dual of the iterated bar construction $B^n A$ (the E_n-algebra structure on $\mathbb{D}(A)$ is dual to the n commuting coalgebra structures on $B^n A$ resulting from the bar construction). Using this mechanism, one can reduce the proofs of many statements about Koszul duality to the case where $n = 1$. For example, one can prove assertions $(K1)$ through $(K3)$ using this method.

8. The Proof of Theorem 6.20

Let k be a field and $n \geq 0$ an integer, fixed throughout this section. Our goal is to prove Theorem 6.20, which asserts the existence of an equivalence of ∞-categories $\Phi : \mathrm{Moduli}_n \rightarrow \mathrm{Alg}^{(n)}_{\mathrm{aug}}$. We begin by describing the inverse to the functor Φ.

Construction 8.1. We let $\Psi : \mathrm{Alg}^{(n)}_{\mathrm{aug}} \rightarrow \mathrm{Fun}(\mathrm{Alg}^{(n)}_{\mathrm{sm}}, \mathcal{S})$ be the functor determined by the formula

$$\Psi(A)(B) = \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}B, A).$$

Here $\mathbb{D} : (\mathrm{Alg}^{(n)}_{\mathrm{aug}})^{op} \rightarrow \mathrm{Alg}^{(n)}_{\mathrm{aug}}$ is the Koszul duality functor of §7.

Suppose we are given a pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

of small E_n-algebras over k, with $\pi_0 A \rightarrow \pi_0 B$ and $\pi_0 B' \rightarrow \pi_0 B$ surjective. Using $(K3)$, we deduce that for every augmented E_n-algebra C over k, we have a pullback square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}A', C) & \longrightarrow & \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}A, C) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}B', C) & \longrightarrow & \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}B, C). \end{array}$$

in the ∞-category $\mathcal{S}$. In other words, the functor $\Psi(C) : \mathrm{Alg}^{(n)}_{\mathrm{sm}} \rightarrow \mathcal{S}$ is a formal E_n moduli problem over k, in the sense of Definition 4.6 (note that $\Psi(C)(k) \simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}(k), C) \simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(k, C)$ is contractible). We may therefore view Ψ as a functor from $\mathrm{Alg}^{(n)}_{\mathrm{aug}}$ to the ∞-category Moduli_n.

Proposition 8.2. *Let A be an augmented E_n-algebra over k. Then there is an equivalence of k-module spectra $\mathfrak{m}_A[n] \simeq T_{\Psi(A)}$, which depends functorially on A.*

Proof. According to Remark 5.1, it will suffice to construct an equivalence between the excisive functors $U, U' : \mathrm{Mod}_{\mathrm{sm}} \rightarrow \mathcal{S}$ given by the formulas

$$U(V) = \mathrm{Hom}_k(V^\vee, \mathfrak{m}_A[n]) \qquad U'(V) = \mathrm{Hom}_k(V^\vee, T_{\Psi(A)}).$$

Let V be a small k-module spectrum. The pairing $(k \oplus V) \wedge_k \mathrm{Free}(V^\vee[-n])$ of $(K1)$ gives rise to a map $\theta_V : \mathrm{Free}(V^\vee[-n]) \rightarrow \mathbb{D}(k \oplus V)$. We obtain maps

$$\begin{array}{rcl} U'(A) & = & \mathrm{Hom}_k(V^\vee, T_{\Psi(A)}) \\ & \simeq & \Psi(A)(k \oplus V) \\ & \simeq & \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}(k \oplus V), A) \\ & \xrightarrow{\theta_V} & \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathrm{Free}(V^\vee[-n]), A) \\ & \simeq & \mathrm{Hom}_k(V^\vee[-n], \mathfrak{m}_A) \\ & \simeq & \mathrm{Hom}_k(V^\vee, \mathfrak{m}_A[n]) \\ & = & U(A). \end{array}$$

To complete the proof, it will suffice to show that θ_V is an equivalence. Note that if $V \simeq V' \times V''$, then $(K3)$ allows us to identify $\mathbb{D}(k \oplus V)$ with the product of the augmented E_n-algebras $\mathbb{D}(k \oplus V')$ and $\mathbb{D}(k \oplus V'')$, so that θ_V is a coproduct of the maps $\theta_{V'}$ and $\theta_{V''}$. Using this observation repeatedly, we can reduce to proving that θ_V is an equivalence when V is a k-module spectrum of the form $k[m]$ for $m \geq 0$, which follows from $(K2)$. □

To complete the proof of Theorem 6.20, it will suffice to construct a functor $\Phi : \mathrm{Moduli}_n \to \mathrm{Alg}^{(n)}_{\mathrm{aug}}$ which is homotopy inverse to Ψ. We will begin by constructing Φ as a left adjoint to Ψ, and later show that this left adjoint is actually an inverse.

Definition 8.3. Let $X \in \mathrm{Moduli}_n$ be a formal E_n moduli problem over k, and let A be an augmented E_n-algebra over A. We will say that a natural transformation $\alpha : X \to \Psi(A)$ *reflects* X if, for every augmented E_n-algebra B over k, composition with α induces a homotopy equivalence

$$\mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(A, B) \to \mathrm{Hom}_{\mathrm{Moduli}_n}(X, \Psi(B)).$$

We let $\mathrm{Moduli}^{\mathrm{o}}_n$ denote the full subcategory of Moduli_n spanned by those formal E_n-module problems X for which there exists a map $\alpha : X \to \Psi(A)$ which reflects F. In this case, the map $\alpha : X \to \Psi(A)$ is well-defined up to canonical equivalence; in particular, we can regard the construction $X \mapsto A$ as defining a functor

$$\Phi : \mathrm{Moduli}^{\mathrm{o}}_n \to \mathrm{Alg}^{(n)}_{\mathrm{aug}}.$$

The functor Φ is left adjoint to Ψ, in the sense that for every $X \in \mathrm{Moduli}^{\mathrm{o}}_n$ and every $B \in \mathrm{Alg}^{(n)}_{\mathrm{aug}}$, we have a canonical homotopy equivalence

$$\mathrm{Hom}_{\mathrm{Moduli}_n}(X, \Psi(B)) \simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\Phi(X), B).$$

Remark 8.4. Since the functor $\Psi : \mathrm{Alg}^{(n)}_{\mathrm{aug}} \to \mathrm{Moduli}_n$ preserves small limits, one can deduce the existence of a left adjoint to Ψ using the adjoint functor theorem. In other words, it follows formally that $\mathrm{Moduli}^{\mathrm{o}}_n = \mathrm{Moduli}_n$. However, we will establish this equality by a more direct argument, which will help us to compute with the functor Φ.

We begin by describing the behavior of the functor Φ on very simple types of formal moduli problems.

Example 8.5. Let A be an small E_n-algebra over k. We let $\mathrm{Spec}\, A \in \mathrm{Moduli}_n$ denote the representable functor $\mathrm{Alg}^{(n)}_{\mathrm{sm}} \to \mathcal{S}$ given by the formula $(\mathrm{Spec}\, A)(B) = \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{sm}}}(A, B)$. Let A be a small E_n-algebra over k. Then $\mathrm{Spec}\, A \in \mathrm{Moduli}^{\mathrm{o}}_n$, and $\Phi(\mathrm{Spec}_A) \simeq \mathbb{D}(A)$. More precisely, the canonical map

$$\mathrm{Spec}(A)(B) = \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(A, B) \to \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathbb{D}(B), \mathbb{D}(A)) = \Psi(\mathbb{D}A)(B)$$

reflects $\mathrm{Spec}(A)$.

Definition 8.6. If $\mathcal{C}$ is any ∞-category, we let $\mathrm{Pro}(\mathcal{C})$ denote the ∞-category of *pro-objects* of $\mathcal{C}$. The objects of $\mathrm{Pro}(\mathcal{C})$ can be identified with formal filtered limits $\varprojlim C_\alpha$ of objects $C \in \mathcal{C}$, and the morphisms in $\mathrm{Pro}(\mathcal{C})$ are computed by the formula

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\varprojlim C_\alpha, \varprojlim D_\beta) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D_\beta).$$

Remark 8.7. There is a parallel theory of Ind-*objects* of ∞-categories: if $\mathcal{C}$ is an ∞-category, then one can define a new ∞-category $\mathrm{Ind}(\mathcal{C})$ by the formula $\mathrm{Ind}(\mathcal{C})^{op} = \mathrm{Pro}(\mathcal{C}^{op})$.

Definition 8.8. If $A \simeq \varprojlim A_\alpha$ is a pro-object of $\mathrm{Alg}^{(n)}_{\mathrm{sm}}$, we let $\mathrm{Spf}(A) : \mathrm{Alg}^{(n)}_{\mathrm{sm}} \to \mathcal{S}$ denote the functor given by the formula

$$B \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Alg}^{(n)}_{\mathrm{sm}})}(A, B) \simeq \varinjlim \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{sm}}}(A_\alpha, B).$$

Remark 8.9. If $A \simeq \varprojlim A_\alpha$ is a pro-object of $\mathrm{Alg}^{(n)}_{\mathrm{sm}}$, then $\mathrm{Spf}\, A \simeq \varinjlim \mathrm{Spec}\, A_\alpha$; it follows immediately that $\mathrm{Spf}\, A \in \mathrm{Moduli}_n$. It follows from Example 8.5 that $\mathrm{Spf}\, A \in \mathrm{Moduli}^{\mathrm{o}}_n$, and that $\Phi(\mathrm{Spf}\, A)$ can be identified with the direct limit

$$\varinjlim \Phi(\mathrm{Spec}\, A_\alpha) \simeq \varinjlim \mathbb{D}(A_\alpha).$$

Remark 8.10. The ∞-category $\mathrm{Alg}^{(n)}_{\mathrm{sm}}$ has a final object (namely, the algebra k) and admits fiber products. It follows by general nonsense that the construction $A \mapsto \mathrm{Spf}\, A$ determines a fully faithful embedding of $\mathrm{Pro}(\mathrm{Alg}^{(n)}_{\mathrm{sm}})$ into $\mathrm{Fun}(\mathrm{Alg}^{(n)}_{\mathrm{sm}}, \mathcal{S})$, whose essential image is the collection of *left exact* functors $X : \mathrm{Alg}^{(n)}_{\mathrm{sm}} \to \mathcal{S}$: that is, functors X such that $X(k)$ is contractible and X carries fiber products in $\mathrm{Alg}^{(n)}_{\mathrm{sm}}$ to fiber products in $\mathcal{S}$. Any left-exact functor is obviously a formal E_n moduli problem.

Not every formal E_n-moduli problem is of the form $\mathrm{Spf}\, A$. For example, it is not difficult to see that if $A \in \mathrm{Pro}(\mathrm{Alg}^{(n)}_{\mathrm{sm}})$ then the space $(\mathrm{Spf}\, A)(k[\epsilon]/(\epsilon^2))$ is homotopy discrete: that is, the homotopy groups $\pi_i \mathrm{Spf}(A)(k[\epsilon]/(\epsilon^2)) \simeq \pi_i T_{\mathrm{Spf}\, A}$ vanish for $i > 0$. However, we can *resolve* every formal E_n-moduli problem X by formal E_n moduli problems of the form $\mathrm{Spf}\, A$.

Lemma 8.11. *Let $\alpha : X \to X'$ be a map of formal E_n moduli problems over k. The following conditions are equivalent:*

1. *The induced map of vector spaces $\pi_i T_X \to \pi_i T_{X'}$ is surjective for $i = 0$ and bijective for $i < 0$.*
2. *Let $A \to B$ be a map of small E_n-algebras over k which induces a surjection $\pi_0 A \to \pi_0 B$. Then the induced map $\pi_0 X(A) \to \pi_0(X(B) \times_{X'(B)} X'(A))$ is surjective.*

Definition 8.12. We say that a map $\alpha : X \to X'$ is *smooth* if it satisfies the equivalent conditions of Lemma 8.11. We say that a formal moduli problem X is *smooth* if the projection $X \to *$ is smooth: that is, if the homotopy groups $\pi_i T_X$ vanish for $i < 0$.

Proof of Lemma 8.11. Let $f : A \to B$ be a map of small E_n-algebras which induces a surjection $\pi_0 A \to \pi_0 B$. Claim 6.14 implies that f factors as a composition

$$A = A(0) \to A(1) \to \cdots \to A(m) = B,$$

where each $A(i) \to A(i+1)$ fits into a pullback diagram

$$\begin{array}{ccc} A(i) & \longrightarrow & A(i+1) \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[j+1] \end{array}$$

for some $j \geq\geq 0$. Consequently, to prove (2), it suffices to treat the case where $A = k$ and $B = k \oplus k[j+1]$. In this case, the space $\pi_0 F(A)$ is contractible, so condition (2) is equivalent to the requirement that $X(B) \times_{X'(B)} X(A)$ is connected. We have a long exact sequence

$$\pi_{-j} T_X \to \pi_{-j} T_{X'} \to \pi_0(X(B) \times_{X'(B)} X(A)) \to \pi_{-j-1} T_X \to \pi_{-j-1} T_{X'},$$

so that $X(B) \times_{X'(B)} X(A)$ is connected if and only if the map $\pi_{-j} T_X \to \pi_{-j} T_{X'}$ is surjective and the map $\pi_{-j-1} T_X \to \pi_{-j-1} T_{X'}$ is injective. These conditions hold for all $j \geq 0$ if and only if α satisfies (1). □

Remark 8.13. Taking $B = k$ in part (2) of Lemma 8.11, we deduce that every smooth map $\alpha : X \to X'$ of formal moduli problems induces a *surjection* $\pi_0 X(A) \to \pi_0 X'(A)$, for every small E_n-algebra A.

Lemma 8.14. *Let $A \in \mathrm{Pro}(\mathrm{Alg}^{(n)}_{\mathrm{aug}})$ and let $\gamma : \mathrm{Spf}\, A \to X$ be a map of formal E_n moduli problems. Then γ can be written as a composition*

$$\mathrm{Spf}\, A \to \mathrm{Spf}\, B \xrightarrow{\beta} F$$

where β is smooth.

Proof. We follow the formal skeleton of Schlessinger's construction (see [27]). We define a tower of pro-objects

$$\cdots \to A(2) \to A(1) \to A(0)$$

and a compatible sequence of maps $\operatorname{Spf} A(i) \xrightarrow{\gamma_i} X$, where $A(0) = A$ and $\gamma_0 = \gamma$. Assume that $A(i)$ has been constructed, and let S denote the collection of all isomorphism classes of diagrams

$$\begin{array}{ccc} \operatorname{Spec} B_\alpha & \longrightarrow & \operatorname{Spec} A_\alpha \\ \downarrow & & \downarrow{\scriptstyle \theta_\alpha} \\ \operatorname{Spf} A(i) & \longrightarrow & X \end{array}$$

such that the map $\pi_0 A_\alpha \to \pi_0 B_\alpha$ is surjective. We now define $A(i+1) = A(i) \times_{\prod_\alpha B_\alpha} \prod_\alpha A_\alpha$; here the products and fiber products are formed in the ∞-category $\operatorname{Pro}(\operatorname{Alg}^{(n)}_{\mathrm{sm}})$. One shows that the maps γ_i and θ_α amalgamate to define a map $\gamma_{i+1} : A(i+1) \to X$. Passing to the limit we get a map $\beta : \operatorname{Spf} B \to X$, which is easily shown to be smooth. □

To proceed further, we need a bit of simplicial technology. Let $\mathbf{\Delta}_+$ denote the category whose objects are finite sets $[m] = \{0, 1, \ldots, m\}$ for $m \geq -1$, and whose morphisms are nondecreasing maps $[m] \to [m']$. We let $\mathbf{\Delta}$ denote the full subcategory spanned by the objects $[n]$ for $n \geq 0$. If $\mathcal{C}$ is any ∞-category, an *simplicial object* of $\mathcal{C}$ is a functor $X_\bullet : \mathbf{\Delta}^{op} \to \mathcal{C}$. If $X_\bullet$ is a simplicial object of $\mathcal{C}$, we will denote the image of $[n] \in \mathbf{\Delta}^{op}$ by X_n, and the colimit of the diagram $X_\bullet$ by $|X_\bullet|$ (if such a colimit exists); we refer to $|X_\bullet|$ as the *geometric realization* of $X_\bullet$.

Remark 8.15. The formation of geometric realizations of simplicial objects is an example of a *sifted* colimit. As such, the formation of geometric realizations tends to be compatible with algebraic structures. For example, the forgetful functor $\operatorname{Alg}^{(n)}_k \to \operatorname{Mod}_k$ commutes with geometric realizations. Similarly, the augmentation ideal functor $\operatorname{Alg}^{(n)}_{\mathrm{aug}} \to \operatorname{Mod}_k$, given by $A \mapsto \mathfrak{m}_A$, commutes with geometric realizations.

An *augmented simplicial object* is a functor $\overline{X}_\bullet : \mathbf{\Delta}^{op}_+ \to \mathcal{C}$. In this case, we will denote the underlying simplicial object $\overline{X}_\bullet | \mathbf{\Delta}^{op}$ by $X_\bullet$. Note that since $[-1] = \emptyset$ is an initial object of $\mathbf{\Delta}_+$, giving an augmented simplicial object $\overline{X}_\bullet$ of $\mathcal{C}$ is equivalent to giving the underlying simplicial object $X_\bullet$, together with another object $X = \overline{X}_{-1}$ equipped with a compatible family of maps $\{X_n \to X\}_{[n] \in \mathbf{\Delta}}$. If $\mathcal{C}$ admits small colimits, then we can identify this family of maps with a single map $|X_\bullet| \to X$.

Let $\overline{X}_\bullet$ be an augmented simplicial object of an ∞-category $\mathcal{C}$ which admits finite limits. For every integer $m \geq 0$, the *mth matching object* $M_m(\overline{X}_\bullet)$ is defined to be the limit $\varprojlim X_{m'}$, where the limit is taken over all proper inclusions $[m'] \hookrightarrow [m]$ (equivalently, the limit is taken over all proper subsets of $\{0, 1, \ldots, m\}$). For each $m \geq 0$, there is a canonical map $X_m \to M_m(X_\bullet)$.

We will need the following fact from simplicial homotopy:

Proposition 8.16. *Let $\overline{X}_\bullet : \mathbf{\Delta}^{op}_+ \to \mathcal{S}$ be an augmented simplicial space. Suppose that, for each $m \geq 0$, the map $\pi_0 X_m \to \pi_0 M_m(\overline{X}_\bullet)$ is surjective. Then the augmentation map $|X_\bullet| \to X = \overline{X}_{-1}$ is a homotopy equivalence.*

Definition 8.17. Let $\overline{F}_\bullet$ be an augmented simplicial object of Moduli_n. We will say that $F_\bullet$ is a *smooth hypercovering* if, for each $m \geq 0$, the map of formal E_n moduli problems $F_m \to M_m(\overline{F}_\bullet)$ is smooth. In this case, we will say that the underlying simplicial object $F_\bullet$ is a *smooth hypercovering* of the formal E_n moduli problem $\overline{F}_{-1}$.

Remark 8.18. Combining Lemma 8.13 with Proposition 8.16, we deduce that if $F_\bullet$ is a a smooth hypercovering of $F \in \operatorname{Moduli}_n$, then the induced map $|F_\bullet(A)| \to F(A)$ is a homotopy equivalence for every small E_n-algebra A. In particular, we conclude that the augmentation map $|F_\bullet| \to F$ is an equivalence of formal E_n moduli problems over k. Similarly, the augmentation map $|T_{F_\bullet}| \to T_F$ is an equivalence of k-module spectra.

By repeatedly applying Lemma 8.14, we obtain the following:

Proposition 8.19. *Let X be a formal E_n moduli problem. Then there exists a smooth hypercovering $X_\bullet$ of X, such that each X_m has the form $\operatorname{Spf} A^m$ for some pro-object $A^m \in \operatorname{Pro}(\operatorname{Alg}^{(n)}_{\mathrm{sm}})$.*

Corollary 8.20. *Every formal E_n moduli problem X belongs to* Moduli_n^o. *Moreover, we can write* $\Phi(X) \simeq |\Phi(\mathrm{Spf}\, A^\bullet)|$*, where* $A^\bullet$ *is as in Proposition 8.19.*

This completes the construction of the functor $\Phi : \mathrm{Moduli}_n \to \mathrm{Alg}^{(n)}_{\mathrm{aug}}$ appearing in Theorem 6.20. By construction, we have an adjunction

$$\mathrm{Moduli}_n \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Alg}^{(n)}_{\mathrm{aug}}.$$

We wish to prove that the functors Φ and Ψ are mutually inverse equivalences of ∞-categories. Our next step is to prove the following:

Proposition 8.21. *The functor* $\Phi : \mathrm{Moduli}_n \to \mathrm{Alg}^{(n)}_{\mathrm{aug}}$ *is fully faithful.*

Proof. We must show that the unit transformation $u : \mathrm{id}_{\mathrm{Moduli}_n} \to \Psi \circ \Phi$ is an equivalence of functors. In other words, we claim that for every formal E_n moduli problem X, the unit map $u_F : X \to (\Psi \circ \Phi)(X)$ is an equivalence of formal moduli problems. Proposition 8.2 implies that the tangent complex $T_{(\Psi\circ\Phi)(X)}$ can be identified with the shifted augmentation ideal $\mathfrak{m}_{\Phi(X)}[n]$ of the augmented E_n-algebra $\Phi(X)$. Passing to tangent complexes, u_F gives a map of k-module spectra $\gamma_X : T_X \to \mathfrak{m}_{\Phi(X)}[n]$. By Proposition 6.23, it will suffice to prove that γ_X is an equivalence. We consider several cases:

(1) Suppose first that $X = \mathrm{Spec}\, A$ for some small E_n-algebra A. Then $\Phi(F) = \mathbb{D}(A)$ (Example 8.5). We wish to prove that the canonical map $\gamma_{\mathrm{Spec}\, A} : T_{\mathrm{Spec}\, A} \to \mathfrak{m}_{\mathbb{D}A}[n]$ is an equivalence of k-module spectra. As in Remark 5.1, it suffices to show that for every small k-module spectrum V, the induced map $\gamma_V : \mathrm{Hom}_k(V^\vee, T_{\mathrm{Spec}\, A}) \to \mathrm{Hom}_k(V^\vee, \mathfrak{m}_{\mathbb{D}(A)}[n])$ is a homotopy equivalence. It now suffices to observe that γ_V is given by the composition

$$\begin{aligned}
\mathrm{Hom}_k(V^\vee, T_{\mathrm{Spec}\, A}) &\simeq (\mathrm{Spec}\, A)(k \oplus V) \\
&\simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(A, k \oplus V) \\
&\simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(A, \mathbb{D}\,\mathrm{Free}(V^\vee[-n])) \\
&\simeq \mathrm{Hom}_{\mathrm{Alg}^{(n)}_{\mathrm{aug}}}(\mathrm{Free}(V^\vee[-n]), \mathbb{D}A) \\
&\simeq \mathrm{Hom}_k(V^\vee[-n], \mathfrak{m}_{\mathbb{D}A}) \\
&\simeq \mathrm{Hom}_k(V^\vee, \mathfrak{m}_{\mathbb{D}A}[n]).
\end{aligned}$$

(2) Now suppose that $X = \mathrm{Spf}\, A$ for some pro-object $A = \varprojlim A_\alpha \in \mathrm{Pro}(\mathrm{Alg}^{(n)}_{\mathrm{sm}})$. We wish to prove that the canonical map $\gamma_{\mathrm{Spf}\, A} : T_{\mathrm{Spf}\, A} \to \mathfrak{m}_{\Phi(\mathrm{Spf}\, A)}$ is an equivalence. This follows immediately from (1), since $\gamma_{\mathrm{Spf}\, A}$ is a filtered colimit of the maps $\gamma_{\mathrm{Spec}\, A_\alpha}$.

(3) Let $X : \mathrm{Alg}^{(n)}_{\mathrm{sm}} \to \mathcal{S}$ be an arbitrary formal E_n moduli problem. By Proposition 8.19, we can choose a smooth hypercovering $X_\bullet$ of X, where each $X_m \simeq \mathrm{Spf}\, A^m$ for some pro-object $A^m \in \mathrm{Pro}(\mathrm{Alg}^{(n)}_{\mathrm{sm}})$. We have a commutative diagram

$$\begin{array}{ccc}
|T_{X_\bullet}| & \xrightarrow{\gamma'} & |\mathfrak{m}_{\Phi X_\bullet}[n]| \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\
T_X & \xrightarrow{\gamma_F} & \mathfrak{m}_{\Phi X}[n].
\end{array}$$

Remark 8.18 implies that α is an equivalence, and Remark 8.15 implies that β is an equivalence. Since γ' is an equivalence by case (2), we deduce that γ_X is an equivalence as desired.

□

We are now ready to prove our main result.

Proof of Theorem 6.20. We have already constructed a fully faithful embedding $\Phi : \mathrm{Moduli}_n \to \mathrm{Alg}^{(n)}_{\mathrm{aug}}$ which admits a right adjoint Ψ, such that the augmentation ideal functor $U : \mathrm{Alg}^{(n)}_{\mathrm{aug}} \to \mathrm{Mod}_k$ is given by $A \mapsto T_{\Psi(A)}[n]$ (Proposition 8.2). To complete the proof, it will suffice to show that Φ is essentially surjective.

This is equivalent to the assertion that Ψ is *conservative*: that is, that a map $\alpha : A \to B$ of augmented E_n-algebras is an equivalence if and only if $\Psi(\alpha) : \Psi(A) \to \Psi(B)$ is an equivalence. This is clear: if $\Psi(\alpha)$ is an equivalence, then we deduce from Proposition 8.2 that α induces an equivalence of augmentation ideals $\mathfrak{m}_A \to \mathfrak{m}_B$ and is therefore itself an equivalence. □

Remark 8.22. It is possible to prove Theorem 5.3 using the same method outlined in this section. The arguments are essentially the same, and make use of the fact that when k is a field of characteristic zero, there exists a Koszul duality functor $\mathbb{D} : (\mathrm{CAlg}_{\mathrm{sm}})^{op} \to \mathrm{Lie}_k^{\mathrm{dg}}$ satisfying analogues of conditions $(K1)$, $(K2)$, and $(K3)$.

9. Examples: Deformation of Objects and Categories

In this section, we will illustrate Theorem 6.20 by considering examples of formal moduli problems drawn from deformation theory.

Definition 9.1. Let $\mathcal{C}$ be an ∞-category which admits small colimits. We say that $\mathcal{C}$ is *stable* if the following conditions are satisfied:

- (i) The ∞-category $\mathcal{C}$ contains an object 0 which is both initial and final.
- (ii) The suspension functor $X \mapsto \Sigma(X) = 0 \coprod_X 0$ is an equivalence of ∞-categories from $\mathcal{C}$ to itself.

We let SCat denote the ∞-category whose objects are stable ∞-categories which admit small colimits, and whose morphisms are functors which preserve small colimits.

Remark 9.2. If $\mathcal{C}$ is a stable ∞-category, then one can show that the homotopy category $\mathrm{h}\mathcal{C}$ is triangulated. In particular, for every pair of objects $C, C' \in \mathcal{C}$, one can define Ext-groups $\mathrm{Ext}^n_{\mathcal{C}}(C, C') = \pi_0 \operatorname{Hom}_{\mathcal{C}}(C, \Sigma^n C')$.

Example 9.3. Let $R \in \mathrm{Alg}^{(1)}(\mathrm{Sp})$ be an E_1-ring. Then the ∞-category $\mathrm{Mod}_R = \mathrm{Mod}_R(\mathrm{Sp})$ is stable, and can be regarded as an object of SCat.

Remark 9.4. The ∞-category SCat admits a symmetric monoidal structure. Roughly speaking, if $\mathcal{C}, \mathcal{D} \in$ SCat, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is characterized by the following universal property: for every object $\mathcal{E} \in$ SCat, giving a colimit-preserving functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is equivalent to giving a bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserves colimits separately in each variable.

Remark 9.5. The construction $R \mapsto \mathrm{Mod}_R(\mathrm{Sp})$ of Example 9.3 is symmetric monoidal: that is, we have $\mathrm{Mod}_R(\mathrm{Sp}) \otimes \mathrm{Mod}_{R'}(\mathrm{Sp}) \simeq \mathrm{Mod}_{R \wedge R'}(\mathrm{Sp})$. Consequently, it defines a functor

$$\mathrm{Alg}^{(n)}(\mathrm{Sp}) \simeq \mathrm{Alg}^{(n-1)}(\mathrm{Alg}^{(1)}(\mathrm{Sp})) \to \mathrm{Alg}^{(n-1)}(\mathrm{SCat}).$$

In particular, if R is an E_2-ring, then the ∞-category $\mathrm{Mod}_R(\mathrm{Sp})$ can be regarded as an associative algebra in the ∞-category SCat: that is, $\mathrm{Mod}_R(\mathrm{Sp})$ is an example of a *monoidal* ∞-category.

Definition 9.6. Let $R \in \mathrm{Alg}^{(2)}(\mathrm{Sp})$ be an E_2-ring. An *R-linear ∞-category* is a $\mathrm{Mod}_R(\mathrm{Sp})$-module object of SCat: that is, a stable ∞-category $\mathcal{C}$ which admits small colimits and is equipped with a coherently associative action

$$\mathrm{Mod}_R(\mathrm{Sp}) \times \mathcal{C} \to \mathcal{C}$$

which preserves small colimits separately in each variable. The collection of all R-linear ∞-categories is itself organized into an ∞-category, which we will denote by SCat_R.

In order to guarantee that an R-linear ∞-category is well-behaved, it is useful to introduce a finiteness assumption.

Definition 9.7. Let $\mathcal{C}$ be an ∞-category which admits small colimits. We say that an object $C \in \mathcal{C}$ is *compact* if the functor $D \mapsto \operatorname{Hom}_{\mathcal{C}}(C, D)$ commutes with filtered colimits. We say that $\mathcal{C}$ is *compactly generated* if there exists a set S of compact objects in $\mathcal{C}$ such that every object of $\mathcal{C}$ can be obtained as a filtered colimit of objects belonging to S.

Notation 9.8. Let R be an E_2-ring. We let $\mathrm{SCat}_R^{\mathrm{cg}}$ denote the full subcategory of SCat_R spanned by those R-linear ∞-categories which are compactly generated.

Remark 9.9. Let $f : R \to R'$ be a map of E_2-rings. Then f induces a functor $\mathrm{SCat}_R \to \mathrm{SCat}_{R'}$, given by $\mathcal{C} \mapsto \mathcal{C}_{R'} = \mathrm{SCat}_{R'} \otimes_{\mathrm{SCat}_R} \mathcal{C}$. Concretely, we can identify $\mathcal{C}_{R'}$ with the ∞-category of *R'-module objects of* $\mathcal{C}$: that is, objects $M \in \mathcal{C}$ equipped with an action $R' \otimes M \to M$ (which is associative up to coherent homotopy).

If $\mathcal{C}$ is compactly generated, then the ∞-category $\mathcal{C}_{R'}$ is also compactly generated: it has compact generators of the form $R' \otimes_R M$, where M ranges over a set of compact generators for $\mathcal{C}$.

Remark 9.10. Let k be a field, regarded as a discrete E_2-ring. Then giving a compactly generated k-linear ∞-category is equivalent to giving a small differential graded category of k, which is well-defined up to Morita equivalence. We refer the reader to [29] for an exposition of the theory of differential graded categories.

We are now ready to introduce a deformation problem:

Definition 9.11. Let k be a field and let $\mathcal{C}$ be a k-linear ∞-category. For every object $C \in \mathcal{C}$, we define a functor $\mathrm{Def}(C) : \mathrm{Alg}^{(1)}_{\mathrm{sm}} \to \mathcal{S}$ as follows. For every small E_1-algebra R, we let $\mathrm{Def}(C)(R)$ denote the fiber (over the object C) of the functor

$$\mathrm{Mod}_R(\mathrm{Sp}) \otimes_{\mathrm{Mod}_k(\mathrm{Sp})} \mathcal{C} = \mathrm{Mod}_R(\mathcal{C}) \to \mathrm{Mod}_k(\mathcal{C}) \simeq \mathcal{C}\,.$$

That is, $\mathrm{Def}(C)(R)$ is the ∞-groupoid of pairs $(\widetilde{C}, \eta)$, where $\widetilde{C}$ is an R-module object of $\mathcal{C}$ and $\eta : k \otimes_R \widetilde{C} \simeq C$ is an equivalence.

Remark 9.12. Efimov, Lunts, and Orlov have made an extensive study of a variant of the deformation functor $\mathrm{Def}(C)$ of Definition 9.11. We refer the reader to [5], [6], and [7] for details. The global structure of moduli spaces of objects of (well-behaved) differential graded categories is treated in [30].

For a general object $C \in \mathcal{C}$, the functor $\mathrm{Def}(C) : \mathrm{Alg}^{(1),}_{\mathrm{sm}} \to \mathcal{S}$ need not be a formal E_1 moduli problem in the sense of Definitino 6.16. However, some mild assumptions on C and $\mathcal{C}$ will guarantee that this is indeed the case:

Proposition 9.13. *Let k be a field, let $\mathcal{C}$ be a compactly generated k-linear ∞-category, and let $C \in \mathcal{C}$ be an object. Suppose that the following condition is satisfied:*

(∗) *For every compact object $X \in \mathcal{C}$, the groups $\mathrm{Ext}^n_{\mathcal{C}}(X, C)$ vanish for $n \gg 0$.*

Then $\mathrm{Def}(C)$ is a formal E_1 moduli problem over k.

Remark 9.14. Under the hypotheses of Proposition 9.13, Theorem 6.20 asserts the formal E_1-moduli problem $\mathrm{Def}(C)$ is “controlled” by an augmented E_1-algebra $A = \Phi(\mathrm{Def}(C))$, or equivalently by the augmentation ideal $\mathfrak{m}_A$ (viewed as a nonunital E_1-algebra over k). One can show that $\mathfrak{m}_A$ is equivalent to the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(C)$; in particular, the tangent complex to $\mathrm{Def}(C)$ can be described by the formula $\pi_n T_{\mathrm{Def}(C)} \simeq \mathrm{Ext}^{1-n}_{\mathcal{C}}(C, C)$.

We now discuss a categorification of the previous moduli problem: rather than deforming a single object $C \in \mathcal{C}$ while keeping the ∞-category $\mathcal{C}$ fixed, we deform the entire ∞-category $\mathcal{C}$.

Definition 9.15. Let k be a field, and let $\mathcal{C}$ be a compactly generated k-linear ∞-category. We define a functor $\mathrm{Def}(\mathcal{C}) : \mathrm{Alg}^{(2)}_{\mathrm{sm}} \to \mathcal{S}$ as follows: to every small E_2-algebra R over k, we let $\mathrm{Def}(\mathcal{C})$ denote the fiber product $\mathrm{SCat}^{\mathrm{cg}}_R \times_{\mathrm{SCat}^{\mathrm{cg}}_k} \{\mathcal{C}\}$. In other words, $\mathrm{Def}(\mathcal{C})(R)$ is the ∞-groupoid whose objects are pairs $(\widetilde{\mathcal{C}}, \alpha)$, where $\widetilde{\mathcal{C}}$ is a compactly generated R-linear ∞-category and $\alpha : \mathcal{C} \simeq \widetilde{\mathcal{C}}_k$ is an equivalence of k-linear ∞-categories.

Remark 9.16. For a more extensive discussion of the deformation theory of differential graded categories, we refer the reader to [16].

As in the previous discussion, we need some hypotheses to guarantee that $\mathrm{Def}(\mathcal{C})$ is a formal moduli problem in the sense of Definition 6.16. The following criterion will be sufficient for the application that we describe in §10:

Proposition 9.17. *Let k be a field and let $\mathcal{C}$ be a compactly generated k-linear ∞-category. Assume that there exists a set of compact objects $\{C_\alpha\}_{\alpha \in A}$ for $\mathcal{C}$ with the following properties:*

(1) *The objects C_α generate $\mathcal{C}$ in the following sense: if $C \in \mathcal{C}_0$ is such that the abelian groups $\mathrm{Ext}^n_{\mathcal{C}}(C_\alpha, C)$ vanish for all $\alpha \in A$ and all integers n, then $C \simeq 0$.*
(2) *For each $\alpha \in A$, the groups $\mathrm{Ext}^n_{\mathcal{C}}(C_\alpha, C_\alpha)$ vanish for $n \geq 2$.*
(3) *For every pair $\alpha, \beta \in A$, the groups $\mathrm{Ext}^n_{\mathcal{C}}(C_\alpha, C_\beta)$ vanish for $n \gg 0$.*

Then the functor $\mathrm{Def}(\mathcal{C}) : \mathrm{Alg}^{(2)}_{\mathrm{sm}} \to \mathcal{S}$ of Definition 9.15 is a formal E_2 moduli problem over k.

Remark 9.18. According to Proposition 9.13, condition (3) of Proposition 9.17 implies that each of the objects $C_\alpha \in \mathcal{C}$ defines a formal E_1-module problem $\mathrm{Def}(C_\alpha)$. In view of Remark 9.14, conditions (2) is equivalent to the requirement that the formal moduli problems $\mathrm{Def}(C_\alpha)$ are *smooth* (Definition 8.12). This can be used to show that each C_α can be lifted to a compact object in any (compactly generated) deformation of the k-linear ∞-category $\mathcal{C}$.

Remark 9.19. Let $\mathcal{C}$ be as in Proposition 9.17. According to Theorem 6.20, the formal E_2 moduli problem $\mathrm{Def}(\mathcal{C})$ is determined up to equivalence by the augmented E_2-algebra $A = \Phi\,\mathrm{Def}(\mathcal{C})$; equivalently, $\mathrm{Def}(\mathcal{C})$ is determined by the nonunital E_2-algebra $\mathfrak{m}_A$. One can show that $\mathfrak{m}_A$ can be identified with the endomorphism ring $\mathrm{End}_{\mathcal{D}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$, where $\mathcal{D}$ is the stable ∞-category of k-linear functors from $\mathcal{C}$ to itself. In particular, we conclude that the tangent complex of $\mathrm{Def}(\mathcal{C})$ is described by the formula

$$\pi_n T_{\mathrm{Def}(\mathcal{C})} \simeq \mathrm{H}^{2-n}(\mathcal{C}),$$

where $\mathrm{H}^m(\mathcal{C}) \simeq \mathrm{Ext}^m_{\mathcal{D}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$ denotes the mth Hochschild cohomology group of $\mathcal{C}$.

Variation 9.20. Let k be a field and let $\mathcal{C} \in \mathrm{SCat}^{\mathrm{cg}}_k$ be a compactly generated k-linear ∞-category. Suppose that $\mathcal{C}$ has the structure of an E_n-algebra object of $\mathrm{SCat}^{\mathrm{cg}}_k$: that is, $\mathcal{C}$ is equipped with n coherently associative tensor product operations $\{\otimes_i : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\}_{1 \leq i \leq n}$ which commute with one another and preserve colimits separately in each variable. We can the define a deformation problem $\mathrm{Def}^{(n)}(\mathcal{C}) : \mathrm{Alg}^{(n+2),}_{\mathrm{sm}} \to \mathcal{S}$, where for every small E_{n+2}-algebra R over k, we let $\mathrm{Def}^{(n)}(\mathcal{C})(R)$ denote the fiber of the map $\mathrm{Alg}^{(n)}(\mathrm{SCat}^{\mathrm{cg}}_R) \to \mathrm{Alg}^{(n)}(\mathrm{SCat}^{\mathrm{cg}}_k)$ over the object $\mathcal{C}$. That is, $\mathrm{Def}^{(n)}(\mathcal{C})$ assigns to R the ∞-groupoid of deformations of $\mathcal{C}$ over R, as an E_n-monoidal ∞-category. (The assumption that R is an E_{n+2}-algebra is needed to guarantee that $\mathrm{SCat}^{\mathrm{cg}}_R$ itself inherits an E_n-monoidal structure).

Suppose moreover that $\mathcal{C}$ satisfies the hypotheses of Proposition 9.17, together with the following additional condition:

($*$) The unit object $\mathbf{1} \in \mathcal{C}$ is compact, and the collection of compact objects of $\mathcal{C}$ is stable under tensor products.

Then one can show that $\mathrm{Def}^{(n)}(\mathcal{C})$ is a formal E_{n+2}-moduli problem over k. It follows from Theorem 6.20 that $\mathrm{Def}^{(n)}(\mathcal{C})$ is determined by an augmented E_{n+2}-algebra $A = \Phi(\mathrm{Def}^{(n)}(\mathcal{C}))$, whose augmentation ideal $\mathfrak{m}_A$ can be described as the fiber of the a map $\mathrm{End}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}) \to \mathrm{End}_{\mathcal{C}}(\mathbf{1})$, where $\mathcal{Z}(\mathcal{C})$ denotes a generalized Drinfeld center of $\mathcal{C}$.

Example 9.21. Let k be a field and $\mathcal{C}$ be a compactly generated k-linear ∞-category satisfying the hypotheses of Proposition 9.17. Fix a compact object $C \in \mathcal{C}$; we can then regard $\mathcal{C}$ as an E_0-algebra object of $\mathrm{SCat}^{\mathrm{cg}}_k$, whose unit object $\mathbf{1}$ is given by C. The analysis of Variation 9.20 (in the case $n = 0$) defines a formal E_2 moduli problem $\mathrm{Def}^{(0)}(\mathcal{C}) : \mathrm{Alg}^{(2),}_{\mathrm{sm}} \to \mathcal{S}$, whose tangent complex $T_{\mathrm{Def}^{(0)}(\mathcal{C})}$ fits into a fiber sequence

$$T_{\mathrm{Def}^{(0)}(\mathcal{C})}[-2] \to \mathrm{End}_{\mathrm{Fun}_k(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \to \mathrm{End}_{\mathcal{C}}(C, C).$$

In particular, we have a long exact sequence

$$\cdots \to \mathrm{Ext}^{1-n}_{\mathcal{C}}(C, C) \to \pi_n T_{\mathrm{Def}^{(0)}(\mathcal{C})} \to \mathrm{Ext}^{2-n}_{\mathrm{Fun}_k(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \to \mathrm{Ext}^{2-n}_{\mathcal{C}}(C, C) \to \cdots$$

which arises from a fiber sequence of moduli problems $\mathrm{Def}(C) \to \mathrm{Def}^{(0)}(\mathcal{C}) \to \mathrm{Def}(\mathcal{C})$; see Remarks 9.14 and 9.19. For a related discussion, we refer the reader to [19] and [20].

Remark 9.22. For a discussion of the deformation theory of monoidal categories over discrete commutative k-algebras, we refer the reader to [3]. The deformation theory of braided monoidal categories is discussed in [34]. See also [8].

10. Deformations of Representation Categories

Our goal in this section is to give a terse sketch how some of the ideas introduced in §9 play out for the deformation theory of the representation category of a reductive algebraic group G. This is the subject of ongoing joint work with Dennis Gaitsgory.

Fix a field k and let G be a reductive algebraic group over k. Let $\mathcal{C}$ denote the ordinary category whose objects are bounded chain complexes of finite-dimensional representations of G, and let W be the collection of all morphisms $w : V_\bullet \to W_\bullet$ in $\mathcal{C}$ which induce isomorphisms $\mathrm{H}_*(V) \to \mathrm{H}_*(W)$ on homology. We let $\mathrm{Rep}(G)$ denote the ∞-category $\mathrm{Ind}(\mathcal{C}[W^{-1}])$ (see Remark 8.7). We will refer to $\mathrm{Rep}(G)$ as the *∞-category of algebraic representations of G.*

Remark 10.1. If the field k has characteristic zero, then the representation theory of G is semisimple and $\mathrm{Rep}(G)$ admits a simpler description: it can be identified with the ∞-category obtained from the category of unbounded chain complexes of possibly infinite-dimensional representations of G by inverting quasi-isomorphisms.

The ∞-category $\mathrm{Rep}(G)$ is k-linear (in the sense of Definition 9.6) and compactly generated. In fact, the irreducible representations of G constitute a collection of compact generators for $\mathrm{Rep}(G)$ which satisfy the hypotheses of Proposition 9.17. Moreover, the formation of tensor products of representations endows $\mathrm{Rep}(G)$ with a symmetric monoidal structure. Consequently, for any $n \geq 0$, we can apply Variation 9.20 to obtain a formal E_{n+2} moduli problem $\mathrm{Def}^{(n)}(\mathrm{Rep}(G))$.

Example 10.2. The vector space $\pi_0 T_{\mathrm{Def}^{(2)}(\mathrm{Rep}(G))}$ can be identified with the set of isomorphism classes of braided monoidal deformations of the (ordinary) category of representations of G over the ring $k[\epsilon]/(\epsilon^2)$. Such deformations were classified by Drinfeld: if G is simple and k is of characteristic zero, then $\pi_0 T_{\mathrm{Def}^{(2)}(\mathrm{Rep}(G))}$ is a one-dimensional vector space over k, generated by a class corresponding to the quantum deformation of G.

Remark 10.3. The algebraic group G over k admits a canonical split form $G_{\mathbf{Z}}$ over the commutative ring $\mathbf{Z}$. Replacing k by $\mathbf{Z}$ in the above discussion, we obtain a $\mathbf{Z}$-linear ∞-category $\mathrm{Rep}(G)_{\mathbf{Z}}$ In §3, we raised the question of whether or not it is possible to do better: for example, can one define a form of G over the sphere spectrum S? As a first step, one can ask if there exists an S-linear ∞-category $\mathcal{C}$ such that $\mathrm{Mod}_{\mathbf{Z}}(\mathcal{C}) \simeq \mathrm{Rep}(G)_{\mathbf{Z}}$. Since the sphere spectrum S can be realized as the limit of a tower of "square-zero" extensions of E_∞-rings

$$\cdots \to \tau_{\leq 2} S \to \tau_{\leq 1} S \to \tau_{\leq 0} S \simeq \mathbf{Z},$$

questions regarding the existence and uniqueness of $\mathcal{C}$ can be attacked using methods of deformation theory. This highlights the importance of understanding formal moduli problems of the form $\mathrm{Def}^{(n)}(\mathrm{Rep}(G))$.

Over a field of characteristic zero, it is not difficult to explicitly describe the deformation problem $\mathrm{Def}^{(n)}(\mathrm{Rep}(G))$ for any n (for example, by computing its tangent complex as a nonunital E_{n+2}-algebra, using methods similar to those described in [8]). However, we will specialize to the case $n = 2$ and adopt a different approach, using ideas from geometric representation theory. Let k be the field $\mathbf{C}$ of complex numbers and let $G^\vee$ denote the Langlands dual group of G, regarded as a reductive algebraic group over $\mathbf{C}$. The quotient $G^\vee(\mathbf{C}((t)))/G^\vee(\mathbf{C}[[t]])$ is called the *affine Grassmannian* for the group $G^\vee$. Following ideas introduced in [10], one can define an ∞-category $\mathrm{Whit}(\mathrm{Gr}_{G^\vee})$ of *Whittaker sheaves* on $\mathrm{Gr}_{G^\vee}$. The following result is essentially proven in [10]:

Theorem 10.4 (Frenkel-Gaitsgory-Vilonen)**.** *There is an equivalence of* $\mathbf{C}$*-linear* ∞*-categories* $\mathrm{Rep}(G) \simeq \mathrm{Whit}(\mathrm{Gr}_{G^\vee})$.

Remark 10.5. Let $BG^\vee$ denote the classifying space of the topological group $G^\vee(\mathbf{C})$. Then the quotient $\mathrm{Gr}_{G^\vee}$ is homotopy equivalent to the two-fold loop space $\Omega^2 BG^\vee$, and therefore has the structure of an E_2-algebra in the ∞-category $\mathcal{S}$. This structure is reflected algebraically in the existence of a *fusion product* on the ∞-category of Whittaker sheaves $\mathrm{Whit}(\mathrm{Gr}_{G^\vee})$, which endows $\mathrm{Whit}(\mathrm{Gr}_{G^\vee})$ with the structure of an E_2-algebra in $\mathrm{SCat}_{\mathbf{C}}$. The equivalence $\mathrm{Rep}(G) \simeq \mathrm{Whit}(\mathrm{Gr}_{G^\vee})$ can be promoted to an equivalence of E_2-algebras: that is, the braided monoidal structure on $\mathrm{Whit}(\mathrm{Gr}_{G^\vee})$ given by the fusion product can be identified with the braided monoidal structure on $\mathrm{Rep}(G)$, given by tensor products of representations.

One can use Theorem 10.4 to produce deformations of the ∞-category $\mathrm{Rep}(G)$. To explain this, we need to embark on a bit of a digression.

Let R be an E_n-ring for $n > 0$. Then the 0th space $\Omega^\infty R$ of R is an E_n-algebra in the ∞-category of spaces: in particular, the set of connected components $\pi_0 \Omega^\infty R \simeq \pi_0 R$ has the structure of a monoid. We let $R^\times \subseteq \Omega^\infty R$ denote the union of those connected components corresponding to invertible elements of the monoid $\pi_0 R$. Then $R^\times$ is a grouplike E_n-algebra, so Example 6.3 supplies a pointed $(n-1)$-connected space Z such that $R^\times \simeq \Omega^n Z$. We will denote the space Z by $\Omega^{-n} R^\times$.

Definition 10.6. Let R be an E_n-ring for $n \geq 2$ and let X be a topological space. An *R-gerbe* on X is a map of topological spaces $X \to \Omega^{-2} R^\times$.

Remark 10.7. When R is a discrete commutative ring, an R-gerbe on X can be identified with a map from X into an Eilenberg-MacLane space $K(R^\times, 2)$. Homotopy classes of R-gerbes are classified by the cohomology group $\mathrm{H}^2(X; R^\times)$.

If $\eta : X \to \Omega^{-2} R^\times$ is an R-gerbe on a topological space X, then there is an associated theory of *η-twisted* sheaves of R-module spectra on X. In the particular case $X = \mathrm{Gr}_{G^\vee}$ and R is a small E_2-algebra over $\mathbf{C}$, one can associate to η an ∞-category $\mathrm{Whit}^\eta(\mathrm{Gr}_{G^\vee})$ of η-twisted Whittaker sheaves on $\mathrm{Gr}_{G^\vee}$. However, this ∞-category will not admit a monoidal structure in general. To guarantee that $\mathrm{Whit}^\eta(\mathrm{Gr}_{G^\vee})$ is an E_2-algebra in SCat_R, we must assume that the gerbe η is *multiplicative*: that is, that the map $\mathrm{Gr}_{G^\vee} \to \Omega^{-2} R^\times$ is itself a map of double loop spaces $\mathcal{S}$. This motivates the following:

Definition 10.8. Let R be an E_4-ring. A *multiplicative R-gerbe* on $\mathrm{Gr}_{G^\vee}$ is a map of pointed topological spaces $BG^\vee \to \Omega^{-4} R^\times$ (equivalently, we can define a multiplicative R-gerbe to be a map of E_2-algebras in $\mathcal{S}$ from $\mathrm{Gr}_{G^\vee}$ to $\Omega^{-2} R^\times$). The collection of multiplicative R-gerbes on $\mathrm{Gr}_{G^\vee}$ is parametrized by a space which we will denote by $\mathrm{Gerbe}(R)$.

Remark 10.9. If R is a small E_4-algebra over a field k, we let $\mathrm{Gerbe}_0(R)$ denote the fiber of the map $\mathrm{Gerbe}(R) \to \mathrm{Gerbe}(k)$. The construction $R \mapsto \mathrm{Gerbe}_0(R)$ defines a formal E_4-moduli problem over k. According to Theorem 6.20, the $\mathrm{Gerbe}_0(R)$ is determined by an augmented E_4-algebra, which in this case can be identified with the cochain algebra $C^*(BG^\vee; k)$ (with augmentation given by the base point on the classifying space $BG^\vee$).

If R is a small E_4-algebra over $\mathbf{C}$ and $\eta \in \mathrm{Gerbe}_0(R)$, then η determines a multiplicative gerbe over $\mathrm{Gr}_{G^\vee}$ which can be used to construct an ∞-category of twisted Whittaker sheaves $\mathrm{Whit}^\eta(\mathrm{Gr}_{G^\vee})$. Using the geometry of the affine Grassmannian $\mathrm{Gr}_{G^\vee}$, one can prove the following:

Theorem 10.10. *Let G be a reductive algebraic group over the field $\mathbf{C}$ of complex numbers. Then the construction $\eta \mapsto \mathrm{Whit}^\eta(\mathrm{Gr}_{G^\vee})$ defines an equivalence of formal E_4 moduli problems*

$$\mathrm{Gerbe}_0 \to \mathrm{Def}^{(2)}(\mathrm{Rep}(G)).$$

In particular, the tangent complex $T_{\mathrm{Def}^{(2)}(\mathrm{Rep}(G))}$ is described by the formula

$$\pi_n T_{\mathrm{Def}^{(2)}(\mathrm{Rep}(G))} \simeq \mathrm{H}^{4-n}_{red}(BG^\vee; \mathbf{C}).$$

Remark 10.11. When $n = 0$, the isomorphism $\pi_0 T_{\mathrm{Def}^{(2)}(\mathrm{Rep}(G))} \simeq \mathrm{H}^4(BG^\vee; \mathbf{C})$ recovers Example 10.2, and suggests that the representation theory of the quantum deformation of G can be described as a category of twisted Whittaker sheaves. We refer the reader to [13] for a proof of this assertion.

Remark 10.12. We expect that an analogue of Theorem 10.10 should continue to hold when the ground field $\mathbf{C}$ is replaced by an arbitrary $\mathbf{Z}[q, q^{-1}]$-algebra and the algebraic group G is replaced by Lusztig's quantum group. We will return to this problem elsewhere.

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