# Higher categories student seminar

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# 0 Introduction – Peter Teichner and Chris Schommer-Pries

## 0.1 Part 1: Lower higher categories – Peter

We will give two different motivations for the study of (higher) categories.

### 0.1.1 Motivation 1: Higher categories show up everywhere

Categories just used to be used to group mathematicians into their subject areas; it's only recently that categories are studied in their own right. For instance, to teach linear algebra is to explore the category $\mathtt{VECT}^{f.d.}$. Of course, there's the eternal argument: is this better to work with abstract vector spaces or with matrices? Well in fact, they amount to *the same thing*, by the equivalence of categories $\mathtt{MAT} \simeq \mathtt{VECT}^{f.d.}$, where the former has objects $\mathbb{N}$, and $\mathtt{MAT}(m,n)$ consists of the $(m \times n)$-matrices. Explicitly, this equivalence is given by $n \mapsto K^n$ (for $K$ our chosen base field), but the inverse is noncanonical.

Why do we like $\mathtt{VECT}_K^{f.d.}$ better? Well, in `MAT` the objects (considered as vector spaces) come with chosen bases, but the maps ignore the bases. From a categorical point of view, one would prefer to have maps that preserve whatever structure our objects come equipped with.

But let's go further: notice that we're working over a field $K$. But these themselves form a category, `FIELDS`. These two categories should interact! In fact, we claim that 2-categories have already appeared twice in this lecture:

1. First, the notion of equivalence of categories uses the (strict) 2-category

   $$\mathtt{CAT} = \begin{cases} 2 & \text{natural isomorphisms} \\ 1 & \text{functors} \\ 0 & \text{categories} \end{cases}$$

   Indeed, $\mathtt{VECT}^{f.d.}$ and `MAT` are *not* isomorphic! (For instance, the latter is small, but the former isn't.)

   We note here that 2-morphisms can be composed both "horizontally" and "vertically": if we have functors $C_2 \xleftarrow{G,G'} C_1 \xleftarrow{F,F',F''} C_0$ and natural transformations $(G \overset{\theta}{\Rightarrow} G')^t$, $(F \overset{\eta}{\Rightarrow} F')^t$, and $(F' \overset{\eta'}{\Rightarrow} F'')^t$, then we have the vertical composition $(F \overset{\eta' \circ \eta}{\Rightarrow} F'')^t$ and the horizontal composition $(G \circ F \overset{\theta \circ \eta}{\Rightarrow} G' \circ F')^t$.

2. As long as we're here, let's change from `FIELDS` to `RINGS` (and replace $\mathtt{VECT}^{f.d.}$ with $\mathtt{MOD}_R$). We claim that to truly undestand the structure of linear algebra, one must study the *double category*

   | | |
   |---|---|
   | 2 | intertwiners |
   | $1^{\text{vert.}}$ - $1^{\text{hor.}}$ | ring homomorphisms (with composition) - bimodules (with tensor product) . |
   | 0 | rings |

   Recall that an *intertwiner* sits inside a typical square of morphisms as

   $$\begin{tikzcd} R_1' & R_0' \arrow[l, "M'"'] \\ R_1 \arrow[u, "\varphi_1"] \arrow[ur, phantom, "\Uparrow \eta"] & R_0, \arrow[l, "M"] \arrow[u, "\varphi_0"'] \end{tikzcd}$$

   which in this notation is a homomorphism $\eta : M \to M'$ of abelian groups such that $\eta(r_1 \cdot m \cdot r_0) = \varphi_1(r_1) \cdot \eta(m) \cdot \varphi_0(r_0)$.

   We can obtain from this a (weak) 2-category, as follows. Given the vertical 1-morphism $(R_1 \overset{\varphi}{\leftarrow} R_0)^t$, we turn it into the horizontal 1-morphism ${}_{R_1}R_{1R_0}$ (given the $R_1 - R_0$-bimodule structure in which $R_1$ acts by left multiplication and $R_0$ acts through $\varphi$). Then, composition of ring homomorphisms becomes the tensor product of bimodules: $(R_2 \overset{\varphi'}{\leftarrow} R_1 \overset{\varphi}{\leftarrow} R_0)^t$ turns into $({}_{R_2}R_{2R_1}) \otimes_{R_1} ({}_{R_1}R_{1R_0})$. This gives us the category

   | | |
   |---|---|
   | 2 | intertwiners |
   | 1 | bimodules |
   | 0 | rings. |

   But what's "weak" about this? Well, the "strictness" of `CAT` is referring to the *strict* associativity of composition of 1-morphisms (functors). Here, on the other hand, we only have natural isomorphisms governing associativity – these are called *associators*. But to be honest, we're lucky we have intertwiners around to even be able to say what these should be!

Here is a first step towards understanding weak 2-categories:

$$\text{monoidal categories} \simeq \text{weak 2-categories with one object.}$$

But now we have a 3-category on the board! In what world does the above equivalence hold? 2-categories form a 3-category, so to say this requires 3-categories. And of course, this goes on forever.

### 0.1.2 Motivation 2: The homotopy hypothesis

Suppose that $X$ is a nice (locally path-connected and semilocally simply connected) space, so that it has a universal cover. Here are three classification results, in increasing order of beauty (or slightly more precisely, category-theoretic preferability):

1. If $\pi_0 X = 0$, then

$$\left\{ \begin{array}{ll} 0 & \text{path-connected covering spaces of } X \\ 1 & \text{isomorphisms of spaces over } X \end{array} \right\} \simeq \left\{ \begin{array}{ll} 0 & \text{subgroups of } \pi_1(X, x_0) \\ 1 & \text{isomorphisms induced by conjugation} \end{array} \right\}$$

$$(Y \xrightarrow{p} X) \quad \mapsto \quad p_*(\pi_1(Y, y_0))$$

   for some $y_0 \in p^{-1}(x_0)$. Note that $\pi_1(X, x_0)$ acts on $p^{-1}(x_0)$ by path-lifting, and $p_*(\pi_1(Y, y_0)) \subset \pi_1(X, x_0)$ is precisely the stabilizer of $y_0$. But since $Y$ is connected then the action of $\pi_1(X, x_0)$ is transitive, so this is actually independent of the choice of $y_0$. (The latter category is immediately seen to be equivalent to the category of transitive $\pi_1(X, x_0)$-sets and isomorphisms.)

2. If $\pi_0 X = 0$, then

$$\left\{ \begin{array}{ll} 0 & \text{covering spaces of } X \\ 1 & \text{morphisms of spaces over } X \end{array} \right\} \simeq \pi_1(X, x_0)\text{-sets.}$$

   Here, $\pi_0 Y$ is identified with the set of orbits of the associated $\pi_1(X, x_0)$-set. Note that the latter category is equivalent to the functor category $\mathtt{FUN}(\pi_1(X, x_0) \rightrightarrows *, \mathtt{SET})$.

3. For any space $X$,

$$\left\{ \begin{array}{ll} 0 & \text{covering spaces of } X \\ 1 & \text{morphisms of spaces over } X \end{array} \right\} \simeq \mathtt{FUN}(\pi_{\leq 1} X, \mathtt{SET}),$$

   where

$$\pi_{\leq 1} X = \left\{ \begin{array}{ll} 0 & X \\ 1 & \text{paths up to homotopy rel endpoints} \end{array} \right.$$

   is the *fundamental groupoid* of $X$.

Note that even when $X$ is connected, 3 is still much better than 2. For instance, in equivariant topology one often cannot choose a basepoint. Moreover, one generally uses the van Kampen theorem to compute $\pi_1(X, x_0)$. But this has the annoying (but necessary) hypothesis of connectivity of intersections. On the other hand, the van Kampen theorem for $\pi_{\leq 1} X$ has no such restrictions, and this already makes it more useful.

Let us put this all into a bit more context. The last of the results above fits into the diagram

$$\begin{tikzcd}
\mathtt{GROUPOIDS} \arrow[r, "\sim"] \arrow[d, hook] & \mathtt{1\text{-}TYPE} \arrow[d, hook] \\
\mathtt{CAT} \arrow[r, "B=|N_\bullet|"'] & \mathtt{CW\text{-}COMPLEX.} \arrow[ul, "\pi_{\leq 1}"']
\end{tikzcd}$$

More generally, we would like to have the diagram

$$\begin{tikzcd}
n\text{-}\mathtt{GROUPOIDS} \arrow[r, "\sim"] \arrow[d, hook] & n\text{-}\mathtt{TYPE} \arrow[d, hook] \\
n\text{-}\mathtt{CAT} \arrow[r, "B"'] & \mathtt{CW\text{-}COMPLEX.} \arrow[ul, "\pi_{\leq n}"']
\end{tikzcd}$$

This is the *homotopy hypothesis* (originally due to Grothendieck): any notion of an $n$-category should have a classifying space functor, and this should induce an equivalence (in an appropriate sense!) between $n$-groupoids $n$-types.

We will see that strict 3-groupoids do *not* model all 3-types (e.g. the Postnikov truncation $\tau_{\leq 3}S^2$ of $S^2$). (This 3 is a sharp bound.) So, in the second part of the seminar, we'll study *weak* higher categories.

## 0.2 Part 2: Higher higher categories – Chris

### 0.2.1 Broad outline and motivation

So, now we know that we should only expect the homotopy hypothesis to hold for weak $n$-categories: $n$-types $\simeq$ $n$-groupoids. In the limit $n \to \infty$, this becomes CW-complexes $\simeq$ $\infty$-groupoids; this is another version of the homotopy hypothesis, which is built directly into the foundations of many of the most popular notions of higher categories.

As a matter of convention, we will say that an $(N, n)$-category is an $N$-category where all $k$-morphisms are invertible for $k > n$. Here, $N \leq \infty$. For instance, an $(\infty, 0)$-category is an $\infty$-category with all morphisms invertible above level 0. In other words this is an $\infty$-*groupoid*, i.e. (what should be) a space.

In this part of the seminar, we'll look at models of $(\infty, n)$-categories that have the homotopy hypothesis built in, notably Rezk's $\Theta_n$-spaces. This ends up making the theory very closely related to the usual homotopy theory of CW-complexes, and we will study one particular incarnation of this: the *Baez-Dolan stabilization hypothesis*. This is a higher-categorical version of the Freudenthal suspension theorem, which says that if $X$ is $(k-1)$-connected, then $\pi_i X \to \pi_i \Omega\Sigma X$ is an isomorphism for $i \leq 2(k-1)$. As a corollary, this implies that if $X$ is also an $(n+k)$-type with $k \geq n+2$, then $X \to \Omega\Sigma X$ induces an equivalence of $(n+k)$-types (i.e. $X \simeq \tau_{n+k}\Omega\Sigma X$). In other words, there is a $((k+1)-1)$-connected $(n+k+1)$-type whose loopspace is equivalent to $X$, namely $Y = \Sigma X$. In other words, $X$ can be *canonically* delooped!

At $n = 0$ (so with $k \geq 2$), this is the theory of Eilenberg-MacLane spaces $K(G, k)$. In general, if $G$ is a group, then we can form $K(G, 1) = BG$, which has $n = 0$ and $k = 1$. But this doesn't satisfy our bound (since $k = 1 \not\geq n + 2 = 2$), so we can't deloop further in general. Of course, if $BG \simeq \Omega Y$ then $G = \pi_1 BG = \pi_1 \Omega Y = \pi_2 Y$, so $G$ must be abelian. Bu in fact, this is the only obstruction: if $G$ is abelian, then $K(G, 2)$ has $n = 0$ and $k = 2$, and the bound is satisfies; indeed, $K(G, 2) \simeq \Omega K(G, 3)$, and we can continue all the way up.

The *stabilization hypothesis* is the analog of this stabilization phenomenon in higher category theory. To explain this, we look at the *periodic table of* $(k-1)$-*connected* $(n+k, n+k)$-*categories*:

| | $n = 0$ | $n = 1$ | $n = 2$ |
|---|---|---|---|
| $k = 0$ | sets | categories | 2-categories |
| $k = 1$ | monoids | monoidal categories | monoidal 2-categories |
| $k = 2$ | commutative monoids | braided monoidal categories | braided monoidal 2-categories |
| $k = 3$ | “ ” | symmetric monoidal categories | sylleptic monoidal 2-categories |
| $k = 4$ | “ ” | “ ” | symmetric monoidal 2-categories |
| $k = 5$ | “ ” | “ ” | “ ” |

(This of course extends to the right, but one can actually use "negative thinking" to extend it to the left as well. We won't dabble in such silly games, however.) Let's explain the $n = 0$ column. First, of course a $(-1)$-connected $(0, 0)$-category is just a set. Then, a 0-connected $(1, 1)$-category is just a 1-category with one object; this is exactly the data of a monoid. Then, a 1-connected $(2, 2)$-category is a 2-category with one object and one 1-morphism. The 2-morphisms admit both horizontal and vertical composition, but by the Eckmann-Hilton argument (the same argument which shows that $\pi_2$ is abelian), one can show that these must both be commutative and must agree. So, our 2-morphisms exactly give us the data of a *commutative* monoid. One can check that this column stabilizes after this.

So, the *stabilization hypothesis* posits that the $t$-fold "loop" functor

$$\{\text{(pointed) } (k+t-1)\text{-connected } (n+k+t, n+k+t)\text{-categories}\} \to \{\text{(pointed) } (k-1)\text{-connected } (n+k, n+k)\text{-categories}\}$$

should be an equivalence whenever $k \geq n + 2$.

### 0.2.2 Detailed outline

So, we'll now outline the plan for the second part of the seminar more carefully.

- **Talks 5-6:** These will concern generalities on $(\infty, n)$-categories, as well as some particular models.
- **Talks 7-8:** These will concern the *$E_n$-stabilization hypothesis.* Recall that in spaces, if $X$ is a $(k-1)$-connected $(k+n)$-type with $k \geq n+2$, then $X \simeq \Omega Y$ (where $Y$ will be a $k$-connected $((k+1)+n)$-type). Given any $(k-1)$-connected $X$, however, we can form $Z = \Omega^{k-1}X$. This gives an equivalence between $(k-1)$-connected spaces and certain $(k-1)$-fold loopspaces. Note that $Z$ is an $n$-type with $k \geq n+2$, and so if also $X \simeq \Omega Y$, then $Z$ is actually a $k$-fold loopspace. One uses the *$E_n$-operad* to keep track of $n$-fold deloopings; an "$E_n$-space" is exactly a space with operations like those of an $n$-fold loopspace. (For instance, at $n = 1$ we get the little 1-disks operad.)

  Now, since we built the homotopy hypothesis into our $(\infty, n)$-categories, we can talk about "$E_k$-$(N, n)$-categories" (for $N \leq \infty$), and the forgetful functor

  $$E_k\text{-}(n, n)\text{-categories} \to E_{k-1}\text{-}(n, n)\text{-categories}$$

  will be an equivalence if $k \geq n + 2$. This will follow from certain topological facts (namely the connectivity of configuration spaces of points in $\mathbb{R}^n$) about the $E_n$-operads. The theory will come in Talk 7, and these facts will be proved in Talk 8.
- **Talks 9-11:** These will connect up $E_k$-$(\infty, n)$-categories with pointed $(n + k + 1)$-categories.
- **Talk 12:** This will be about applications to TFT's, given by either Peter or Chris.

# 1 Strict $n$-categories and their classifying spaces – Lars Borutzky

In this talk, all our categories will be *small*; this will allow us to avoid any set-theoretic issues.

## 1.1 Strict $n$-categories, left Kan extensions, and nerves of 1-categories

We begin with the following inductive definition.

**Definition 1.** A *(strict) 0-category* is a set. Then, a *(strict) $n$-category* $\mathcal{C}$ consists of:

- a collection of objects,
- for each pair of objects $x, y$, a strict $(n-1)$-category $\mathcal{C}(x, y)$, and

- composition functors $\mathcal{C}(x,y) \times \mathcal{C}(y,z) \to \mathcal{C}(x,z)$,

such that composition is associative, and for each object $x$ there is an identity object $1_x$ in $\mathcal{C}(x,x)$ that behaves appropriately. (We will make this notion more precise in a bit; we should really have "identity" morphisms all the way up.) Inductively, we say that the 0-*morphisms* of $\mathcal{C}$ are its objects, and the $k$-*morphisms* of $\mathcal{C}$ are the $(k-1)$-morphisms of the various $(n-1)$-categories $\mathcal{C}(x,y)$.

**Example 1.** Let $X$ be a space. Then we can form a 2-category as follows: the objects are the points of $X$, the 1-morphisms from $x$ to $y$ are paths $\gamma : [0,t] \to X$ (for some $t \geq 0$) such that $\gamma(0) = x$ and $\gamma(1) = y$, and the 2-morphisms are homotopy classes of homotopies rel endpoints. (We allow these to run over trapezoids, so that we don't only have morphisms between paths defined on the same interval.) Composition is defined by concatenation; note that this is *strictly* associative. This is the reason we've chosen to use intervals of arbitrary length, rather than defining all our paths on the unit interval.

We will define classifying spaces via *nerve* functors, which come from the following very general framework.

Suppose $\mathcal{C}$ is a cocomplete category, and that we have a functor $F : \mathbf{\Delta} \to \mathcal{C}$. Then we have the *left Kan extension* $L$ of $F$ along the standard inclusion (really the Yoneda embedding) $j : \mathbf{\Delta} \hookrightarrow \texttt{sSet}$, which admits a right adjoint $R$: these all sit in the diagram

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[d, "F"'] \arrow[dr, "j"] & \\
\mathcal{C} \arrow[r, dashed, shift right, "R"'] & \texttt{sSet} \arrow[l, shift right, "L"']
\end{tikzcd}$$

in which the solid arrow commutes (up to natural isomorphism). (This is generally written with `sSet` in the top-right, but our diagram package is too shoddy to be able to pile diagonal arrows.)

The functor $R$ is easy to describe. If $c \in \mathcal{C}$, then $Rc \in \texttt{sSet}$ has $(Rc)_n = \mathcal{C}(F[n], c)$, and for any $f \in \mathbf{\Delta}([\mathbf{n}], [\mathbf{m}])$ we have $(Rc)(f) = (Ff)^* : (Rc)_m \to (Rc)_n$.

The construction of $L$ is cool, so we describe it too. If $X \in \texttt{sSet}$ and $c \in \mathcal{C}$, let us write $X_n \cdot c = \coprod_{X_n} c \in \mathcal{C}$. Now, $f \in \mathbf{\Delta}([\mathbf{n}], [\mathbf{m}])$ induces the diagram

$$\begin{tikzcd}
X_m \cdot F[n] \arrow[r, "X_m \cdot F(f)"] \arrow[d, "X(f) \cdot F{[n]}"'] & X_m \cdot F[m] \\
X_n \cdot F[n].
\end{tikzcd}$$

As we range over all morphisms in $\mathbf{\Delta}$, we get a whole bunch of these corners. We say that a *wedge* of (or a *cocone* on) this diagram is an object $c \in \mathcal{C}$ together with maps $\gamma_n : X_n \cdot F[n] \to c$ such that all diagrams

$$\begin{tikzcd}
X_m \cdot F[n] \arrow[r, "X_m \cdot F(f)"] \arrow[d, "X(f) \cdot F{[n]}"'] & X_m \cdot F[m] \arrow[d, "\gamma_m"] \\
X_n \cdot F[n] \arrow[r, "\gamma_n"'] & c
\end{tikzcd}$$

commute. Finally, a *coend* is a universal wedge (i.e. the initial object in the evident category of wedges). This is denoted by $\int^n X_n \cdot F[n]$. We define $LX = \int^n X_n \cdot F[n]$; the behavior of $L$ on morphisms is given by the universal property.

We observe that $L\Delta^n \cong F[n]$ (this is just the computation of $L\Delta^n = \int^m (\Delta^n)_m \cdot F[m]$), and in fact $L \circ j \cong F$, as we have claimed above. This immediately implies that we have natural isomorphisms

$$\mathtt{sSet}(\Delta^n, Rc) \cong (Rc)_n = \mathcal{C}(F[n], c) \cong \mathcal{C}(L\Delta^n, c)$$

(by Yoneda, by definition, and by this observation, respectively). That is, $L \dashv R$. (Of course, given $X \in \mathtt{sSet}$, we have that $X \cong \mathrm{colim}_{\Delta^n \to X} \Delta^n$. This gives a shorter route to this adjunction.)

**Example 2.** Let $\mathcal{C} = \mathtt{Top}$, and let $F : \mathbf{\Delta} \to \mathtt{Top}$ be given by $F([n]) = |\Delta^n| = \Delta_n$, the topological $n$-simplex. Then $(RY)_n = \mathtt{Top}(\Delta_n, Y)$; that is, $R = \mathrm{Sing}_\bullet$, the *simplicial set of singular simplices* functor. In the other direction, $LX = \int^n X_n \cdot \Delta_n = |X|$, the geometric realization.

**Example 3.** In this framework, we can now precisely define the nerve of a category. Let $\mathcal{C} = \mathtt{Cat}$ and $F : \mathbf{\Delta} \to \mathtt{Cat}$, with $F([n])$ the usual category $\underline{n}$ given by the poset $\{0 \to 1 \to \ldots \to n\}$. If $C \in \mathtt{Cat}$, then we have $(RC)_n = \mathtt{Cat}(\underline{n}, C)$, the sequences of $n$ composable arrows in $C$. Thus $R = N$, the *nerve* functor. In the other direction, the functor $L : \mathtt{sSet} \to \mathtt{Cat}$ takes a simplicial set and returns the category whose objects are the vertices, whose morphisms are the edges, with composition relations generated by the 2-simplices. (Technically, $LX = \tau_1 X$, the *1-truncation* of $X$.)

We make the following inductive definitions, which allow us to complete our previous definition.

**Definition 2.** If $\mathcal{C}$ and $\mathcal{D}$ are $n$-categories, then their *product* is the $n$-category $\mathcal{C} \times \mathcal{D}$ whose objects are given by $\mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$, and with $(\mathcal{C} \times \mathcal{D})((c,d),(c',d')) = \mathcal{C}(c,c') \times \mathcal{D}(d,d')$. Then, an *$n$-functor* $F : \mathcal{C} \to \mathcal{D}$ between $n$-categories is the data of a function $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ and, for each pair $x, y \in \mathcal{C}$, an $(n-1)$-functor $\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$, such that all diagrams

$$\begin{tikzcd}
\mathcal{C}(x,y) \times \mathcal{C}(y,z) \arrow[r] \arrow[d] & \mathcal{D}(Fx,Fy) \times \mathcal{D}(Fy,Fz) \arrow[d] \\
\mathcal{C}(x,z) \arrow[r] & \mathcal{D}(Fx,Fz)
\end{tikzcd}$$

commute.

We can now be more precise about our "identity $k$-morphisms". We define the one-point $(n-1)$-category pt in the obvious way, and then we require for each object $x$ in $\mathcal{C}$ an $(n-1)$-functor $1_x : \mathrm{pt} \to \mathcal{C}(x,x)$ such that the composition

$$\mathcal{C}(x,y) \cong \mathrm{pt} \times \mathcal{C}(x,y) \xrightarrow{1_x \times \mathrm{id}} \mathcal{C}(x,x) \times \mathcal{C}(x,y) \to \mathcal{C}(x,y)$$

is the identity $(n-1)$-functor (and such that $1_y$ similarly yields the identity $(n-1)$-functor).

## 1.2 Higher nerve functors

We would like to generalize our nerve construction to arbitrary strict $n$-categories. We begin at the case $n = 2$. Then we define the *nerve* of a 2-category $C$ to be the functor $\mathbf{\Delta^{op}} \to \mathtt{Cat}$ by taking $[n]$ to the category $\mathtt{Cat}(\underline{n}, C)$, whose objects are strings of $n$ composable arrows in $C$ and whose morphisms are given by strings of $n$ horizontally composable 2-morphisms in $C$. (So in particular, for there to be any morphisms between two $\underline{n}$-diagrams in $C$, they must have the same objects.) We can also obtain this by considering $\underline{n}$ as a 2-category with only trivial 2-morphisms, and then we just take $[n]$ to the 1-category of functors $\underline{n} \to C$ of 2-categories. All in all, we get a functor $NC : \mathbf{\Delta^{op}} \to \mathtt{Fun}(\mathbf{\Delta^{op}}, \mathtt{Set})$, which takes $[n]$ to the functor which takes $[k]$ to strings of $k$ composable natural transformations between functors $[n] \to C$. We consider this as a bisimplicial set $NC : \mathbf{\Delta^{op}} \times \mathbf{\Delta^{op}} \to \mathtt{Set}$.

And so of course, in the general case that $C$ is an $n$-category, we get an $n$-fold simplicial set. We can actually rephrase the entire story as coming from the diagram

$$\begin{tikzcd}
\mathbf{\Delta}^n \arrow[d, "F"'] \arrow[dr, "j"] & \\
n\mathtt{Cat} \arrow[r, shift right, dashed, "R"'] & \mathtt{Fun}((\mathbf{\Delta^n})^{\mathbf{op}}, \mathtt{Set}) = \mathtt{s^nSet}. \arrow[l, shift right, "L"']
\end{tikzcd}$$

Now, $L$ should be considered the "free $n$-category" functor, and $R = N$ should be considered as the "nerve of an $n$-category".

We've already mentioned the geometric realization functor $|\cdot| : \mathtt{sSet} \to \mathtt{Top}$. We can extend inductively by using our coend construction iteratively, and indeed we get $|\cdot| : \mathtt{s}^n\mathtt{Set} \to \mathtt{Top}$ as

$$|X| = \int^{k_1} \left( \cdots \left( \int^{k_{n-1}} \left( \int^{k_n} X_{k_n} \cdot \Delta^{k_n} \right)_{k_{n-1}} \cdot \Delta^{k_{n-1}} \right) \cdots \right).$$

In fact, the realization of a bisimplicial set is isomorphic to the diagonal, and by induction this extends to $n$-fold simplicial sets. In any case, we call the end result of all this is the *classifying space* functor $B = |N \cdot | : n\mathtt{Cat} \to \mathtt{Top}$.

## 1.3 Examples of higher categories

We end with Peter attempting to collect a (hopefully long) list of further examples of strict $n$-categories from the audience.

**Example 4.** `CAT`, the category of small categories, is in fact a strict 2-category. The 1-morphisms are functors, and the 2-morphisms are natural transformations.

**Example 5.** We have the 2-category `GROUPS` of groups; the 1-morphisms are the group homomorphisms, and the 2-morphisms between $\varphi, \varphi' : G_1 \to G_2$ are the elements $g \in G_2$ such that $\varphi' = c_g \circ \varphi$, where $c_g$ denotes conjugation by $g$. Vertical composition comes from multiplication in $G_2$ since $c_{gh} = c_g \circ c_h$, and this is associative because multiplication in $G_2$ is.

This actually embeds into the 2-category `GROUPOIDS`, where the group $G$ becomes the groupoid $G \rightrightarrows *$. In fact, `GROUPS` $\to$ `GROUPOIDS` is a 2-functor. The surprising fact (which we leave as an exercise) is that this is fully faithful: all natural transformations between morphisms of groupoids $(G_1 \rightrightarrows *)$ and $(G_2 \rightrightarrows *)$ come from the 2-morphisms as we have defined above.

**Example 6.** Let $A$ be an abelian monoid. Then there is a strict $n$-category $\mathcal{C}^n_A$, which we can define inductively. This has one object, denoted $*$, and then we set $\mathcal{C}^n_A(*, *) = \mathcal{C}^{n-1}_A$. We begin at $n = 0$ with $\mathcal{C}^0_A = A$. However, we technically only remember that this is a set, so instead we should begin wtih $\mathcal{C}^1_A(*, *) = (A \rightrightarrows *)$. On the other hand, we could instead inductively define $\mathcal{C}^n_A$ as a monoid in $n$-categories, and then we could begin at $n = 0$.

The abelianness becomes necessary at $n = 2$. Namely, $\mathcal{C}^2_A(*, *) = \mathcal{C}^1_A = (A \rightrightarrows *)$, and the composition map $\mathcal{C}^2_A(*, *) \times \mathcal{C}^2_A(*, *) \to \mathcal{C}^2_A(*, *)$ must be a functor. But this is equivalent to asking for the multiplication map $A \times A \to A$ to be a morphisms of monoids, which is true iff $A$ is abelian.

**Example 7. (N.B. that this example is not quite right; we correct it in the next lecture.)** We can consider `TOP` to be a 2-category. One way to do this would be to take the 1-morphisms to be the continuous maps, and the 2-morphisms to be the homotopy classes of homotopies. On the other hand, we could instead take our 2-morphisms to be homotopies indexed by arbitrary intervals $[0, t]$. This recovers $\mathtt{TOP}(*, X)$ as the fundamental 2-category of $X$ that we saw earlier. But now we can finally get an interesting 3-category! Namely, we can take $\mathtt{TOP}$ with $\mathtt{TOP}(A,B)$ the fundamental 2-category of the mapping space space (with the compact-open topology). (This probably requires compactly generated and weak Hausdorff conditions, but only Dave cares about this.) Of course, one should check that $\mathtt{TOP}(A,B) \times \mathtt{TOP}(B,C) \to \mathtt{TOP}(A,C)$ is indeed a 2-functor.

**Example 8.** Whenever we have a topological category (i.e. a category enriched in $\mathtt{TOP}$), then we can get a strict 3-category in this way. This therefore includes simplicial (model) categories, via geometric realization of mapping spaces.

**Example 9.** It is *impossible* to generalize the "Moore paths" construction past 2-categories. This follows from the fact that there exist weak $n$-categories that cannot be strictified for $n \geq 3$.

# 2 Strictification of weak 2-categories – Arik Wilbert

## 2.1 Motivation

Peter begins with a review of the previous lecture, in order to motivate the present one.

Given a category $\mathcal{A}$ with products, we define $\mathcal{A}\text{-}\mathtt{CAT}$ to be the category of categories enriched over $\mathcal{A}$. So for instance, $\mathtt{SET}\text{-}\mathtt{CAT} = \mathtt{CAT} = 1\text{-}\mathtt{CAT}$, $2\text{-}\mathtt{CAT} = \mathtt{CAT}\text{-}\mathtt{CAT}$, and more generally $n\text{-}\mathtt{CAT} = ((n-1)\text{-}\mathtt{CAT})\text{-}\mathtt{CAT}$. Applying classifying spaces take us to $\mathtt{TOP}$ and $\mathtt{TOP}\text{-}\mathtt{CAT}$ respectively, and the nerve functor $|N_\bullet| : \mathtt{TOP}\text{-}\mathtt{CAT} \to \mathtt{TOP}$ makes the diagram commute.

Now, if $F : \mathcal{A} \to \mathcal{B}$ is a product-preserving functor, then we have $\tilde{F} : \mathcal{A}\text{-}\mathtt{CAT} \to \mathcal{B}\text{-}\mathtt{CAT}$. For instance, $i : \mathtt{SET} \to \mathtt{CAT}$ gives us a functor $\tilde{i} : \mathtt{SET}\text{-}\mathtt{CAT} = \mathtt{CAT} \to \mathtt{CAT}\text{-}\mathtt{CAT} = 2\text{-}\mathtt{CAT}$, which takes a 1-category and returns the 2-category which only has identity 2-morphisms. Inductively, we get $n\text{-}\mathtt{CAT} \to (n+1)\text{-}\mathtt{CAT}$.

**Exercise 1.** Show by induction that these functors have both left and right adjoints, and that these adjoints are both product preserving.

**Lemma 1.** *$n\text{-}\mathtt{CAT} = (n\text{-}\mathtt{CAT}, \times)$ has inner Hom's; that is, there are natural isomorphisms (of sets)*

$$n\text{-}\mathtt{CAT}(B \times C, D) \cong n\text{-}\mathtt{CAT}(B, \underline{n\text{-}\mathtt{CAT}}(C,D)).$$

**Corollary 1.** *$n\text{-}\mathtt{CAT}$ is an $(n+1)$-category.*

(This generalizes our previous observation that $\mathtt{CAT}$ is a 2-category.)

In order to prove the lemma, we need the following definition. A *globular set* is given by a diagram of sets of the form

$$A_n \rightrightarrows \cdots \rightrightarrows A_2 \rightrightarrows A_1 \rightrightarrows A_0$$

for some $n < \infty$; the forward maps are called $s$ and $t$ (for "source" and "target"), and there are backwards maps (not pictured), called $u$ (for "unit"); these must of course satisfy certain axioms, the most obvious which being $s \circ u = t \circ u = \mathrm{id}$. Pictorially, one should think of each element of $A_k$ as a "$k$-globe" running between its source and target hemispheres (i.e. $(k-1)$-globes): all of this is just a different way of decomposing the $k$-ball into a CW-complex.

**Lemma 2.** *An $n$-category is a globular set together with multiplications (ending at $A_n$)*

$$\{(a', a) : s^{m-p}(a') = t^{m-p}(a)\} = A_m \times_{A_p} A_m \to A_m$$

*for $0 \leq p < m - 1$, satisfying certain diagrams.*

(Under this correspondence, $A_k$ is the set of $k$-morphisms.)

Now one easily defines $k$-`Cell`, the *free $k$-category on one $k$-morphism*, and then we can define the inner Hom by

$$\underline{n\text{-}\mathtt{CAT}}(C, D)_k = n\text{-}\mathtt{CAT}(i^{n-k}(k\text{-}\mathtt{Cell} \times C, D).$$

For instance, at $n = 2$, $\underline{2\text{-}\mathtt{CAT}}(C, D)$ has objects the functors $C \to D$, 1-morphisms the functors $C \times (\bullet \to \bullet) \to D$ (i.e. natural transformations of functors), and 2-morphisms the functors $C \times (2\text{-}\mathtt{Cell}) \to D$ (where $2\text{-}\mathtt{Cell}$ looks like two parallel arrows and a natural transformation between them).

And now, on to Arik's talk!

## 2.2 The theorem

Today we will be concerned with the following theorem.

**Theorem 1.** *Every bicategory is biequivalent to a 2-category.*

The first half of the talk will explain the ideas in the theorem, and then in the second half we'll sketch a proof.

### 2.2.1 The definitions

**Definition 3.** A *bicategory* $\mathcal{B}$ consists of the following data:

1. a collection of objects $\mathrm{ob}(\mathcal{B})$ (denoted $A, B, C, \ldots$);
2. categories $\mathcal{B}(A, B)$ for all $A, B \in \mathrm{ob}(\mathcal{B})$ (with objects the 1-morphisms $f, g, \ldots$ and morphisms the 2-morphisms $\alpha, \beta, \gamma, \ldots$);
3. *horizontal composition* functors
$$c_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$$
for all $A, B, C \in \mathrm{ob}(\mathcal{B})$ (denoted $(g, f) \mapsto g \circ_1 f$ and $(\beta, \alpha) \mapsto \beta \star \alpha$), along with *identity* functors $I_A : \mathbf{1} \to \mathcal{B}(A, A)$ for all $A \in \mathrm{ob}(\mathcal{B})$ satisfying the usual diagrams;
4. natural isomorphisms
$$\begin{array}{rcl} a & : & c_{ABD} \circ (c_{BCD} \times \mathrm{id}) \to c_{ACD} \circ (\mathrm{id} \times c_{ABC}) \\ l & : & c_{ABB} \circ (I_B \times \mathrm{id}) \to \mathrm{id} \\ r & : & c_{AAB} \circ (\mathrm{id} \times I_A) \to \mathrm{id}, \end{array}$$
called the *associator* and the *left* and *right unitors*, which induce 2-morphisms
$$\begin{array}{l} a_{h,g,f} : (h \circ_1 g) \circ_1 f \to h \circ_1 (g \circ_1 f) \\ l_f : I_B \circ_1 f \to f \\ r_f : f \circ_1 I_A \to f; \end{array}$$

these must satisfy the *pentagon axiom*, dictating that the diagram

$$\begin{tikzcd}
 & (kh)(gf) \arrow[dr, sloped, "a"] & \\
((kh)g)f \arrow[ur, sloped, "a"] \arrow[d, "\alpha\times 1"'] & & k(h(gf)) \\
k(hg)f \arrow[rr, "a"'] & & k((hg)f) \arrow[u, "1\times\alpha"']
\end{tikzcd}$$

commutes, and the *triangle axiom*, dictating that the diagram

$$\begin{tikzcd}
(gI)f \arrow[rr, "a"] \arrow[dr, sloped, "R\star 1"'] & & g(If) \arrow[dl, sloped, "1\star l"] \\
 & gf &
\end{tikzcd}$$

commutes.

**Remark 1.** Note that if $a, l, r$ are identity maps, then $\mathcal{B}$ is just a 2-category. So, one may think of a bicategory as a *weak* 2-category.

**Example 10.** `Cat` is a 2-category, and hence a bicategory.

**Example 11.** A monoidal category is a 1-object bicategory.

**Example 12.** If $\mathcal{B}$ is a bicategory, then we have the dual bicategory $\mathcal{B}^{op}$; this construction reverses 1-morphisms, but keeps the 2-morphisms running in the same direction.

**Example 13.** *Tangles* form a bicategory. The objects are just the nonnegative integers, where $n \geq 0$ is identified with the $n$-point 0-manifold. A 1-morphism is a flat tangle, i.e. a cobordism in $\mathbb{R} \times I \subset \mathbb{R} \times \mathbb{R}$. Then, 2-morphisms are cobordisms up to isotopy rel boundary in $\mathbb{R} \times I \times I$. (If one cares about this, one should look up a picture.)

Now, if we care about bicategories, then we must know what their morphisms are.

**Definition 4.** If $\mathcal{B}$ and $\mathcal{B}'$ are bicategories, then a *homomorphism* of bicategories $(F, \phi) : \mathcal{B} \to \mathcal{B}'$ consists of the following data:

1. a function $F : \mathrm{ob}(\mathcal{B}) \to \mathrm{ob}(\mathcal{B}')$;
2. functors $F_{AB} : \mathcal{B}(A, B) \to \mathcal{B}'(FA, FB)$ (for all $A, B \in \mathrm{ob}(\mathcal{B})$);
3. natural isomorphisms $\phi_{ABC} : c^{\mathcal{B}'}_{FA,FB,FC} \circ (F_{BC} \times F_{AB}) \to F_{AC} \circ c^{\mathcal{B}}_{ABC}$ and $\phi_A : I^{\mathcal{B}'}_{FA} \to F_A \circ I^{\mathcal{B}}_A$ (for all $A, B, C \in \mathrm{ob}(\mathcal{B})$), giving 2-morphisms $\phi_{gf} : Fg \circ_1 \to F(g \circ_1 f)$ and $\phi_A : I^{\mathcal{B}'}_{FA} \to F(I^{\mathcal{B}}_A)$;

these must satisfy certain diagrams that we won't actually write down.

(To see the axioms in full detail, consult e.g. Leinster's paper "Basic Bicategories" or Chris's thesis.)

**Example 14.** We claim that there is a homomorphism of 2-categories $\mathcal{B}(-,A) : \mathcal{B}^{op} \to \mathtt{Cat}$, for any fixed object $A \in \mathrm{ob}(\mathcal{B})$. Given any $B \in \mathrm{ob}(\mathcal{B}^{op})$, we obtain the category $\mathcal{B}(B,A)$. We have

$$\mathcal{B}(-,A)_{BC} : \mathcal{B}^{op}(B,C) \to \mathtt{Cat}(\mathcal{B}(B,A), \mathcal{B}(C,A))$$

by $(B \xrightarrow{f^{op}} C) \mapsto (\mathcal{B}(B,A) \xrightarrow{\mathcal{B}(f^{op},A)_{BC}} \mathcal{B}(C,A))$. But we have a map going backwards, too: if we have a morphism given by $(B \xrightarrow{g} A) \mapsto (C \xrightarrow{f} B \xrightarrow{g} A)$ and $(\alpha : g \Rightarrow h) \mapsto (\alpha \times \mathrm{id} : g \circ_1 f \Rightarrow h \circ_1 f)$, then we can recover the morphism $B \xrightarrow{f^{op}} C$. This example will lead to the *2-Yoneda lemma*, which gives us the *2-Yoneda embedding*.

**Definition 5.** Two bicategories $\mathcal{B}$ and $\mathcal{B}'$ are called *biequivalent* if there exists a homomorphism $F : \mathcal{B} \to \mathcal{B}'$ such that:

1. $F$ is *locally* an equivalence (i.e. $F_{AB}$ is an equivalence of categories for all $A, B \in \mathrm{ob}(\mathcal{B})$);
2. for all $B' \in \mathrm{ob}(\mathcal{B}')$, there exists a object $B \in \mathrm{ob}(\mathcal{B})$ such that $FB$ is internally equivalent. (If $A, B \in \mathrm{ob}(\mathcal{B})$, we say that $A$ and $B$ are *internally equivalent* if there exist 1-morphisms $f : A \leftrightarrows B : g$ together with isomorphisms $(1 \to g \circ_1 f) \in \mathcal{B}(A,A)$ and $(f \circ_1 g \to 1) \in \mathcal{B}(B,B)$.)

### 2.2.2 The proof

We now have all the definitions necessary for the theorem in hand, and so we can embark on the proof itself.

Our goal is, for bicategories $\mathcal{B}$ and $\mathcal{B}'$, to define a bicategory

$$[\mathcal{B}, \mathcal{B}'] = \begin{cases} 2 & \text{modifications} \\ 1 & \text{transformations} \\ 0 & \textit{homomorphisms}. \end{cases}$$

When $\mathcal{B}'$ is a 2-category, this will be a 2-category as well.

Note that if $D \xrightarrow{h} E$ is a 1-morphism in $\mathcal{B}$, then we obtain functors $h_* : \mathcal{B}(C,D) \to \mathcal{B}(C,E)$ and $h^* : \mathcal{B}(E,C) \to \mathcal{B}(D,C)$.

**Definition 6.** A *transformation* (or *strong transformation*) $\sigma : F \to G$ between two homomorphisms $F, G : \mathcal{B} \to \mathcal{B}'$ of bicategories is the following data:

1. 1-morphisms $\sigma_A : FA \to GA$ for all $A \in \mathcal{B}$;
2. natural isomorphisms
$$\sigma_{AB} : (\sigma_A)^* \circ G_{AB} \to (\sigma_B)_* \circ F_{AB}$$
inducing invertible 2-morphisms
$$Gf \circ_1 \sigma_A \to \sigma_B \circ_1 Ff$$
for all 1-morphisms $f : A \to B$ in $\mathcal{B}$;

satisfying certain diagrams that one can look up in the literature.

(Note that the classical definition of a natural transformation involves certain equalities; this is analogous, but we're only using isomorphisms.)

**Definition 7.** A *modification* $\Gamma : \sigma \Rightarrow \tilde{\sigma}$ between two transformations $\sigma, \tilde{\sigma} : F \to G$ of homomorphisms of bicategories $F, G : \mathcal{B} \to \mathcal{B}'$ consists of the following data:

1. a 2-morphism $\Gamma_A : \sigma_A \Rightarrow \tilde{\sigma}_A$ between the 1-morphisms $\sigma_A, \tilde{\sigma}_A : FA \to GA$, for all objects $A \in \mathcal{B}$.

Of course, the analogy runs that bicategories should be thought of as points, homomorphisms should be thought of as directed edges, transformations should be thought of as 2-disks running between two (parallel) directed edges, and modifications should be thought of as 3-balls running between two (parallel) 2-disks.

Let us indicate why $[\mathcal{B},\mathcal{B}']$ is strict when $\mathcal{B}'$ is. We must define the horizontal compositions

$$[\mathcal{B},\mathcal{B}'](G,H)\times[\mathcal{B},\mathcal{B}'](F,G)\to[\mathcal{B},\mathcal{B}'](F,H),$$

which we denote by $(\tilde{\sigma},\sigma)\mapsto\tilde{\sigma}\circ\sigma$. For any object $A\in\mathcal{B}$, this is defined by the commutative diagram

$$\begin{tikzcd}
FA \arrow[rr, "(\tilde{\sigma}\circ\sigma)_A"] \arrow[dr, "\sigma_A"'] & & HA \\
& GA. \arrow[ur, "\tilde{\sigma}_A"'] &
\end{tikzcd}$$

Moreover, for all 1-morphisms $f:A\to B$ in $\mathcal{B}$, if $\mathcal{B}'$ is strict we have the invertible 2-morphism $(\tilde{\sigma}\circ\sigma)_f:Hf\circ_1(\tilde{\sigma}\circ\sigma)_A\to(\tilde{\sigma}\circ\sigma)_B\circ_1 Ff$ given by

$$(Hf\circ_1\tilde{\sigma}_A)\circ_1\sigma_A\xrightarrow{\tilde{\sigma}_f\star\mathrm{id}}(\tilde{\sigma}_B\circ_1 Hf)\circ_1\sigma_A=\tilde{\sigma}_B\circ_1(Hf\circ_1\sigma_A)\xrightarrow{\mathrm{id}\star\sigma_f}\tilde{\sigma}_B\circ_1(\sigma_B\circ_1 Ff)=(\tilde{\sigma}\circ\sigma)_B\circ_1 Ff$$

(where the equalities comes from the assumption that $\mathcal{B}'$ is strict). Now, we can see strict associativity as follows. If $F\xrightarrow{\sigma}G\xrightarrow{\tilde{\sigma}}H\xrightarrow{\tilde{\tilde{\sigma}}}I$ is a sequence of composable 2-morphisms, we must see that $(\tilde{\tilde{\sigma}}\circ\tilde{\sigma})\circ\sigma=\tilde{\tilde{\sigma}}\circ(\tilde{\sigma}\circ\sigma)$. But this follows from unwinding the definitions.

We now present a proof of the main theorem.

*Proof.* Let $\mathcal{B}$ be a bicategory. We consider a homomorphism $Y:\mathcal{B}\to[\mathcal{B}^{op},\mathtt{Cat}]$, which we call the *2-Yoneda embedding.* (Note that the target is a 2-category, since $\mathtt{Cat}$ is strict.) On objects, this is given by an example above, $A\mapsto\mathcal{B}(-,A)$. Now, define the subbicategory $\mathcal{B}'\subset[\mathcal{B}^{op},\mathtt{Cat}]$ to be the *full image* of $\mathcal{B}$, i.e. the full subbicategory spanned by the homomorphisms $YA:\mathcal{B}^{op}\to\mathtt{Cat}$. This defines a homomorphism $Y':\mathcal{B}\to\mathcal{B}'$. This will be our biequivalence; note that $\mathcal{B}'$ is a strict bicategory since it's a subbicategory of a strict one. Of course, this is essentially surjective by construction, s it only remains to check that it's a local equivalence. But this is just the 2-categorical Yoneda lemma that $Y$ is a local equivalence. But this means that $Y'$ is too. (Of course, to actually do this all rigorously, we should think about the 1-morphisms and 2-morphisms; so far, we've only thought about objects. But this is as incredibly messy as it is straightforward, so we forbear.) ☐

The question becomes: Where does this proof essentially fail for tricategories? In fact, if $\mathcal{T}$ is a tricategory, then we will need to consider $\mathcal{T}\to[\mathcal{T}^{op},\mathtt{Bicat}]$, and this target is not strict. The problem is that $\mathtt{Bicat}\to\mathtt{2\text{-}Cat}$ is only a *weak* functor.

We end with a reference: On the nLab, one can search "2-categorical Yoneda lemma", and there is a link to a paper by Igor Baković for a complete hands-on proof. Of course, the nLab itself has a high-powered proof as well.

## 2.3 Correction from previous lecture

Finally, Peter returns to correct something from last time. Recall we had $\pi_{\leq 2}:\mathtt{TOP}\to\mathtt{2\text{-}CAT}$ given by Moore paths (i.e. paths indexed on an interval $[0,l]$ for any $l\geq 0$) and homotopies rel endpoints indexed on trapezoids. (Incidentally we can do this with tangles too, to obtain a strict 2-category.) We concluded wrongly that we can obtain a strict 3-category with objects the compactly generated Hausdorff spaces and with morphisms the 2-categories $\pi_{\leq 2}(\mathtt{TOP}(A,B))$. The problem is that composition is *not* strictly associative.

We do indeed have associative composition maps $\texttt{TOP}(B,C) \times \texttt{TOP}(A,B) \to \texttt{TOP}(A,C)$, but the issue is that $\pi_{\geq 2}$ isn't product-preserving: the problem is the fact that we've got Moore loops, so in a product of spaces we can only naturally take products of paths that have the same length. So we only have the solid diagrams

$$\begin{tikzcd}
\pi_{\leq 2}(\texttt{TOP}(B,C) \times \texttt{TOP}(A,B)) \arrow[rr] \arrow[dr] & & \pi_{\leq 2}(\texttt{TOP}(A,C)) \\
& \pi_{\leq 2}(\texttt{TOP}(B,C)) \times \pi_{\leq 2}(\texttt{TOP}(A,B)), \arrow[ur, dashed] &
\end{tikzcd}$$

and to have strict associativity would require naturally selecting dotted arrows. Now it turns out that the right-downward arrow is in fact an equivalence, so we can choose a section, but this involves a choice which there's no canonical way to make; hence we don't get strict associativity. Thus, we only get a *weak* 3-category.

# 3 2-groupoids and 2-types – Malte Pieper

## 3.1 Motivation

Peter jumps in to give us a review of what we've seen and an idea of where we're headed. Recall that we defined strinct $n$-categories for all $n$, but we saw that this could be generalized. Last week, Arik showed us $\texttt{2-CAT} \subset \texttt{BiCAT}$, and given $B \in \texttt{BiCAT}$, he showed that $Y : B \to \texttt{Fun}(B^{op}, \texttt{CAT})$ is an equivalence onto its essential image. We've defined equivalences in both situations $\texttt{2-CAT}$ and $\texttt{BiCAT}$; thus we can deduce a morphism of homotopy categories: $h\texttt{-2-CAT} \to h\texttt{-BiCAT}$. (This actually might be an equivalence, but there's verbal debate about this. If we have two strict categories and consider the weak functors between them, not all of them can be strictified. This doesn't prove the opposite, but at least it's not transparent that this is an equivalence. Interestingly, it will actually suffice to show that $h\texttt{-2-GRP} \xrightarrow{\sim} h\texttt{-2-types}$.)

Now, remember that Lars showed that there's always a classifying space functor $B : n\texttt{-CAT} \to \texttt{TOP}$, and this respects the inclusions $\texttt{CAT} \to \texttt{1-CAT} \to \cdots \to n\texttt{-CAT}$. Now, the composite functor $\texttt{CAT} \to n\texttt{-CAT} \to \texttt{TOP}$ is already surjective on weak homotopy types. So from the point of view of topology, we don't need the intermediate categories! However, what we will see in more detail today is that we can take a subfiltration of $n\texttt{-GRP} \subset n\texttt{-CAT}$ by $\texttt{GRP} \to \texttt{2-GRP} \to \cdots \to n\texttt{-GRP}$, and this has geometric content: there is a factorization $B : n\texttt{-GRP} \to n\texttt{-types} \subset \texttt{TOP}$. The punchline, then, will be that for $n \leq 2$ these actually induce equivalences of homotopy catories $n\texttt{-GRP} \xrightarrow{\sim} n\texttt{-types}$. However, we will need to weaken our $n$-categories for $n > 2$ in order to get an equivalence here.

And now, on to Malte's talk!

## 3.2 What's a 2-group?

We begin with the definition.

**Definition 8.** A (strictly) *coherent 2-group* is a weak monoidal category with an *adjoint equivalence* for each object, such that all morphisms are invertible. (One can see the precise diagrams on the handout that's been distributed, which is also appended to the end of this section.) These define the category $\texttt{C2G}$. An adjoint equivalence $(g, \overline{g}, i_g, e_g)$ consists of two elements $g, \overline{g} \in \mathcal{G}$ and isomorphisms $i_g : 1 \xrightarrow{\cong} g \otimes \overline{g}$ and $e_g : \overline{g} \otimes g \xrightarrow{\cong} 1$, such that two diagrams commute (which equate maps $1 \otimes g \to g \otimes 1$ and $\overline{g} \otimes 1 \to 1 \otimes \overline{g}$).

**Remark 2.** Now, why do we use this definition of 2-group? What does it have to do with the more intuitive notions that we've already seen? Well, *weak* 2-groups are equivalent to coherent 2-groups by a choice of adjoint equivalences. (A weak 2-group can be defined to be a bicategory with a single object and with all morphisms invertible. Or, we can repeat the definition of coherent 2-group, but replace our requirement of adjoint equivalences by saying that for all $g \in \mathcal{G}$ there is some $\overline{g} \in \mathcal{G}$ such that $g \otimes \overline{g} \cong 1 \cong \overline{g} \otimes g$.)

**Definition 9.** A *homomorphism* of coherent 2-groups is a weak monoidal functor, and a *2-homomorphism* is a weak monoidal natural transformation.

**Remark 3.** Our functor does not explicitly carry over the adjoint equivalences. However, note that it determines an isomorphism $\overline{\mathcal{F}(-)} \xrightarrow{\cong} \mathcal{F}(\overline{-})$ if we claim compatibility with the unit and counit of the adjoint equivalences, as given in the diagrams in the handout.

## 3.3 Classify 2-groups!

In order to begin our classification, we introduce the following definition.

**Definition 10.** A coherent 2-group is called *special* if it is skeletal (i.e. its underlying category is skeletal, i.e. any two isomorphic objects are equal) and the unitors $l$ and $r$ as well as $e$ and $i$ are all identities.

**Proposition 1.** *Every coherent 2-group is equivalent to a special one.*

*Proof sketch.* Essentially the point here is that once we're skeletal, the only issue might be that our isomorphisms are nontrivial automorphisms. It turns out that these can be hidden by fiddling with the 2-morphisms $\mathcal{F}_2 : \mathcal{F}(x) \otimes \mathcal{F}(y) \xrightarrow{\cong} \mathcal{F}(x \otimes y)$, and these can all be turned into equalities at once. ☐

We now introduce an invariant of special 2-groups. This will be the algebraic data necessary to classify them. Given $\mathcal{G}$, we associate the quadruple $(G, H, \alpha, a)$, where:

- $G = (\mathrm{ob}(\mathcal{G}), \otimes)$ is a group because $\mathcal{G}$ is skeletal;
- $H = (\mathrm{Aut}(1_{\mathcal{G}}), \circ)$ is an abelian group (by the Eckmann-Hilton argument);
- $\alpha : G \to \mathrm{Aut}(H)$ given by $g \mapsto (h \mapsto (1_g \otimes h) \otimes 1_{\overline{g}})$ is a group action;
- $a : G^3 \to H$ given by $(g_1, g_2, g_3) \mapsto a_{g_1,g_2,g_3} \otimes 1_{\overline{g_1 \otimes g_2 \otimes g_3}}$ is a normalized cocycle in $C^3(G; H)$, where $H$ is a $\mathbb{Z}[G]$-module via $\alpha$.

(That $a$ is a cocycle follows from the pentagon axiom. To say a cocycle is *normalized* means that if $g_i = 1_G$ for any $i$ then $a(g_1, g_2, g_3) = 1_H$. We recall that *group cohomology* has its $n$-chains given by $C^n(G; H) = \mathrm{Hom}_{\mathtt{Set}}(G^{n+1}, H)$, with e.g. $(\partial a)(g_1, g_2, g_3, g_4) = g_1 \cdot a(g_2, g_3, g_4) - a(g_1 g_2, g_3, g_4) + a(g_1, g_2 g_3, g_4) - a(g_1, g_2, g_3 g_4) + a(g_1, g_2, g_3)$.)

**Proposition 2.** *There is a bijective correspondence between such quadruples* $(G, H, \alpha, a)$ *and special 2-groups up to canonical isomorphism.*

*Proof sketch.* We first indicate how to go back. Given a quadruple $(G, H, \alpha, a)$, we define $\mathcal{G}$ by $\mathrm{ob}(\mathcal{G}) = G$, $\mathcal{G}(g, g) = H$, $\mathcal{G}(g, h) = \emptyset$ for $g \neq h$, and tensor products on morphisms is given by $1_g \otimes - = \alpha(g, -)$ and $- \otimes 1_g = \mathrm{id}$, and hence we set $f \otimes f' = (f \otimes 1) \circ (1 \otimes f')$, and lastly $a_{g_1,g_2,g_3} = a(g_1, g_2, g_3)$.

Now, the composition back to quadruples very obviously is the identity. So, it remains to specify what the canonical isomorphism is. If the composition applied to $\mathcal{G}$ produces $\mathcal{H}$, then we define a canonical isomorphism $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ which is the identity everywhere except on morphisms, where we have $\mathcal{G}(g, g) \to \mathcal{H}(g, g)$ given by $f \mapsto f \otimes 1_{\overline{g}}$. ☐

Now, this is actually a rather weak statement; ideally we'd have an equivalence of categories. So, we give a brief indication of what the morphisms are on the algebraic side. Associated to $\mathcal{F} : \mathcal{G} \to \mathcal{G}'$, the morphism of algebraic quadruples is a triple $(\phi, \psi, k)$, where:

- $\phi : G \to G'$ is a group homomorphism;
- $\psi : H \to H'$ is a homomorphism of $\mathbb{Z}[G]$-modules (i.e. $\psi(\alpha(g,h)) = \alpha(\phi(g), \psi(h))$;
- $k : G^2 \to H'$ given by $k(g_1, g_2) = (\mathcal{F}_2)_{g_1,g_2} \otimes 1_{\overline{g_1 \otimes g_2}}$ is a normalized cochain such that $dk = \psi_*(a) - \phi^*(a')$.

Then, the precise statement is the following.

**Proposition 3.** *We have an equivalence of categories* $\{(G,H,\alpha,a)\}/\text{iso.} \overset{\sim}{\leftrightarrow} \texttt{S2G}/\text{equiv.}$.

Combining with the results of the previous talk, we have the string of equivalences

$$\{(G,H,\alpha,a)\}/\text{iso.} \overset{\sim}{\leftrightarrow} \texttt{S2G}/\text{eq.} \overset{\sim}{\leftrightarrow} \texttt{C2G}/\text{eq.} \overset{\sim}{\leftrightarrow} \texttt{weak 2-groups}/\text{eq.} \overset{\sim}{\leftrightarrow} \text{strict 2-groups/eq.}.$$

Every equivalence here but the last is actually an equivalence of tricategories, but the last one is impossible to strictify in this way.

### 3.4 2-types/equivalence ↔ 2-groups/equivalence

We would like to extend the above to `strict 2-groups`/eq. $\overset{\sim}{\leftrightarrow}$ `connected 2-types`/ $\sim$. To be precise, we make the following definition.

**Definition 11.** A *connected 2-type* $X$ is a CW-complex with exactly one 0-cell, denoted $*$, such that $\pi_i(X, *) = 0$ for all $i > 2$.

We restrict our attention to connected 2-types; we can run this whole machine one connected component at a time, so this is no real loss.

(As an interesting aside, if $X$ and $Y$ are connected 2-types, then it is possible for $\pi_3\text{map}(X, Y) \neq 0$, but it is always the case that $\pi_3\text{map}_*(X, Y) = 0$. As should be clear from our categorical setup, we will only be considering based maps.)

**Definition 12.** We write $\pi_{\leq 2}X$ for the endomorphism category of $* \in X$ in the *path 2-category*, i.e. the 2-category of points of $X$, paths in $X$, and homotopy classes of endpoint-preserving homotopies between paths. So $\pi_{\leq 2}X$ has objects the loops at $*$, and has morphisms the homotopy classes of homotopies rel endpoints.

We give an explicit model for our classifying spaces: $B\mathcal{G} = |N_\bullet(N_\bullet\mathcal{G}, \otimes)|$. (The nerve of $\mathcal{G}$ is levelwise a monoid, and so we can take the nerve again to obtain a bisimplicial set. The geometric realization of this is by definition the classifying space of $\mathcal{G}$.) The element $\varphi \in N_iN_j\mathcal{G}$ (for $i \geq 1$) is equivalent to a string of $j$ composable morphisms running from $g_{1_0} \otimes \cdots \otimes g_{i_0}$ to $g_{1_j} \otimes \cdots \otimes g_{i_j}$.

**Proposition 4.** *$B\mathcal{G}$ is a 2-type. In fact, $B\mathcal{G} = B(|N\mathcal{G}|, \otimes)$; that is, $B\mathcal{G}$ is a model for the classifying space of the usual classifying space $|N\mathcal{G}|$ of $\mathcal{G}$, which is itself a topological monoid with operation induced from $\otimes : N_i\mathcal{G} \times N_i\mathcal{G} \to N_i\mathcal{G}$.*

**Remark 4.** Now, $\pi_{i+1}(B\mathcal{G}) \cong \pi_i(|N_\bullet G|)$. For $j \geq 3$, we claim that we have a filler in the diagram

$$\begin{tikzcd}
\partial\Delta^j \arrow[r] \arrow[d, hook] & N_\bullet\mathcal{G} \arrow[d, two heads] \\
\Delta^j \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

for any horizontal maps making the square commute; this is the statement that $\pi_j(N_\bullet\mathcal{G}) = 0$ for $j \geq 3$. (In general one must bifibrantly replace a simplicial set to compute its simplicial homotopy groups. But $N_\bullet\mathcal{G}$ is a Kan complex (i.e. it is fibrant) since it's the nerve of a groupoid, and moreover every simplicial set is cofibrant. So $N_\bullet\mathcal{G}$ is bifibrant.) This is exactly the statement that $\pi_j(B\mathcal{G}) = 0$ for $j \geq 3$. Now, this diagram is equivalent to asking for a filler in

$$\begin{tikzcd}
\tau_1\partial\Delta^j \arrow[r] \arrow[d] & \mathcal{G} \\
\tau_1\Delta^j. \arrow[ur, dashed] &
\end{tikzcd}$$

(Recall that $\tau_1$ sits in the adjunction $\tau_1 : \texttt{sSet} \leftrightarrows \texttt{Cat} : N$; the objects are the vertices, and the morphisms are generated by the edges modulo the 2-simplices.) But the vertical map is an isomorphism since $\tau_1$ only depends on the 2-skeleton, so in fact there is a *unique* filler.

Now we know that our two constructions land in the right places. So, it remains to show that they are inverses (up to the correct notions of equivalence).

First, let us prove that $\pi_{\leq 2}B\mathcal{G} \simeq \mathcal{G}$. We will have a morphism $\mathcal{F} : \mathcal{G} \to \pi_{\leq 2}B\mathcal{G}$, and passing to special 2-groups induces an isomorphism on the algebraic data $G$ and $H$. Furthermore, we can extend our algebraic data with a cochain to get an inverse in the category of quadruples. So, let us define $\mathcal{F}$. On objects, we take $g \in \mathcal{G}$ to $p_g : I \to B\mathcal{G}$, which is a parametrization of the geometric realization of $g \in N_1N_0\mathcal{G}$. On morphisms, we take $\alpha : g \to h$ to $\hat{\alpha} : H^2 \to B\mathcal{G}$ in the similar way, considering $\alpha \in N_1N_1\mathcal{G}$. One shows that this can be extended to an associative unital functor $\mathcal{F}$; to do this, one must simply guess the right simplices that witness these facts. (For instance, to show that $\mathcal{F}_1$ is a functor, we must show that $\mathcal{F}_1(gh) = \mathcal{F}_1(g) \circ \mathcal{F}_1(h)$. This is witnessed by the evident 2-simplex running from $p_{gh}$ to $p_g \star p_h$, which is associated to $(g, h) \in N_2N_0\mathcal{G}$.) To show that $G$ and $H$ are isomorphisms, we have that $G_{\widetilde{\pi_{\leq 2}B\mathcal{G}}} = (\mathrm{ob}(\widetilde{\pi_{\leq 2}B\mathcal{G}}), \otimes) \cong \pi_1(B\mathcal{G}, *) \cong \pi_0(|N_\bullet\mathcal{G}|) \cong \mathcal{G}_{\widetilde{\mathcal{G}}}$, where the last isomorphism takes $[g]$ to $[p_g]$. (Here, $\widetilde{\mathcal{G}}$ denotes the skeletonization of $\mathcal{G}$.)

Finally, we sketch that $B\pi_{\leq 2}X \simeq X$. For this, we define

$$\begin{tikzcd}
B\pi_{\leq 2}X \arrow[r, "B\pi_{\leq 2}f"] \arrow[d, "t"] & B\pi_{\leq 2}X \arrow[d, "t"] \\
K(G,1) \arrow[r, "f"'] & X
\end{tikzcd}$$

for $f$ a $\pi_1$-isomorphism. One must check that:

1. $\pi_1B\pi_{\leq 2}f$ is an isomorphism;
2. $t$ is natural on $\pi_1$;
3. $t$ is an isomorphism on the left side.

After a similar argument for $\pi_2$, we may apply Whitehead's theorem to obtain that $t$ is a homotopy equivalence.

Now, the first statement amounts to unwinding the definitions and using the above result for the other direction. For the other steps one defines $t$ similarly to the way we defined $\mathcal{F}$ in the previous step and then considers the diagram

$$\begin{tikzcd}
\pi_1(\Omega B\pi_{\leq 2}X) \arrow[r, "\Omega t_*"] & \pi_1(\Omega X) \\
\pi_1(|N_\bullet \pi_{\leq 2}X|) \arrow[u, "\wr"'] \arrow[ur] & \\
\pi_1(|N_\bullet \widetilde{\pi_{\leq 2}X}|), \arrow[u, "\wr"'] &
\end{tikzcd}$$

in which the composition from the bottom to the top right can be explicitly understood for $X = K(G, 1)$.

Chris summarizes: By some hard work, we were able to construct functors in both directions. Going back and forth in both directions admit functorial comparisons, and the point is that this preserves $\pi_1$ and $\pi_2$.

## 3.5 Addendum (handout): Classification of homotopy 2-types: Some basic definitions

**Definition 13.** A coherent 2-group is a category $\mathcal{G}$ together with a bifunctor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, an object $1 \in \mathcal{G}$, natural isomorphisms $l, r$ (unitors) and $a$ (associator) and for every $g \in \mathcal{G}$ an adjoint equivalence $(g, \overline{g}, i_g, e_g)$, s.t. all morphisms are invertible and the following diagrams commute:
Pentagon Identity:

$$\begin{tikzcd}
 & & (g_0 \otimes g_1) \otimes (g_2 \otimes g_3) \arrow[drr, "a_{g_0 g_1 g_2 \otimes g_3}"] & & \\
((g_0 \otimes g_1) \otimes g_2) \otimes g_3 \arrow[urr, "a_{g_0 \otimes g_1 g_2 g_3}"] \arrow[dr, "a_{g_0 g_1 g_2} \otimes 1_{g_3}"'] & & & & g_0 \otimes (g_1 \otimes (g_2 \otimes g_3)) \\
 & (g_0 \otimes (g_1 \otimes g_2)) \otimes g_3 \arrow[rr, "a_{g_0 g_1 \otimes g_2 g_3}"'] & & g_0 \otimes ((g_1 \otimes g_2) \otimes g_3) \arrow[ur, "1_{g_0} \otimes a_{g_1 g_2 g_3}"'] &
\end{tikzcd}$$

Unit Law:

$$\begin{tikzcd}
(g \otimes 1) \otimes h \arrow[rr, "a_{g1h}"] \arrow[dr, "r_g \otimes 1_h"'] & & g \otimes (1 \otimes h) \arrow[dl, "1_g \otimes l_h"] \\
 & g \otimes h &
\end{tikzcd}$$

**Definition 14.** An adjoint equivalence is a quadruple $(g, \overline{g}, i_g, e_g)$, where $g, \overline{g} \in \mathcal{G}$ and $i_g : 1 \to g \otimes \overline{g}$ (unit) and $e_g : \overline{g} \otimes g \to 1$ (counit) are isomorphisms, s.t. the following diagrams commute:

$$\begin{tikzcd}
1 \otimes g \arrow[r, "i_g \otimes 1"] \arrow[d, "l_g"] & (g \otimes \overline{g}) \otimes g \arrow[r, "a_{g\overline{g}g}"] & g \otimes (\overline{g} \otimes g) \arrow[d, "1 \otimes e_g"'] \\
g \arrow[rr, "r_g^{-1}"] & & g \otimes 1
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{g}\otimes 1 \arrow[r, "1\otimes i_g"] \arrow[d, "r_{\overline{g}}"] & (\overline{g}\otimes g)\otimes\overline{g} \arrow[r, "a_{\overline{g}g\overline{g}}"] & \overline{g}\otimes(g\otimes\overline{g}) \arrow[d, "e_{\overline{g}}\otimes 1"] \\
\overline{g} \arrow[rr, "l^{-1}_{\overline{g}}"] & & 1\otimes\overline{g}
\end{tikzcd}$$

**Definition 15.** A homomorphism of coherent 2-groups is a weak monoidal functor $\mathcal{F}$. It consists of a functor $\mathcal{F}_1 : \mathcal{G} \to \mathcal{G}'$, natural isomorphisms $\mathcal{F}_{2_{gh}} : \mathcal{F}_1(g)\otimes\mathcal{F}_1(h) \to \mathcal{F}_1(g\otimes h)$ and an isomorphism $\mathcal{F}_0 : 1' \to \mathcal{F}_1(1)$, s.t. the following diagrams commute:
Compatibility with the associator:

$$\begin{tikzcd}
(\mathcal{F}_1(g)\otimes\mathcal{F}_1(h))\otimes\mathcal{F}_1(k) \arrow[r, "\mathcal{F}_2\otimes 1"] \arrow[d, "a'_{\mathcal{F}_1(g)\mathcal{F}_1(h)\mathcal{F}_1(k)}"] & \mathcal{F}_1(g\otimes h)\otimes\mathcal{F}_1(k) \arrow[r, "\mathcal{F}_2"] & \mathcal{F}_1((g\otimes h)\otimes k) \arrow[d, "\mathcal{F}_1(a_{ghk})"] \\
\mathcal{F}_1(g)\otimes(\mathcal{F}_1(h))\otimes\mathcal{F}_1(k)) \arrow[r, "1\otimes\mathcal{F}_2"] & \mathcal{F}_1(g)\otimes\mathcal{F}_1(h\otimes k) \arrow[r, "\mathcal{F}_2"] & \mathcal{F}_1(g\otimes(h\otimes k))
\end{tikzcd}$$

Compatibility with the unitors:

$$\begin{tikzcd}
1'\otimes\mathcal{F}_1(g) \arrow[r, "l'"] \arrow[d, "\mathcal{F}_0"] & \mathcal{F}_1(g) \\
\mathcal{F}_1(1)\otimes\mathcal{F}_1(g) \arrow[r, "\mathcal{F}_2"] & \mathcal{F}_1(1\otimes g) \arrow[u, "\mathcal{F}_1(l)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_1(g)\otimes 1' \arrow[r, "r'"] \arrow[d, "\mathcal{F}_0"] & \mathcal{F}_1(g) \\
\mathcal{F}_1(g)\otimes\mathcal{F}_1(1) \arrow[r, "\mathcal{F}_2"] & \mathcal{F}_1(g\otimes 1) \arrow[u, "\mathcal{F}_1(r)"']
\end{tikzcd}$$

**Definition 16.** A 2-homomorphism of coherent 2-groups is a weak monoidal natural transformation $\theta : \mathcal{F} \to \mathcal{F}'$, i.e. $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{C}'$ are weak monoidal functors and $\theta$ is a natural transformation $\theta : \mathcal{F}_1 \to \mathcal{F}'_1$, s.t. the following diagrams commute:
Compatibility with $\mathcal{F}_2$ and $\mathcal{F}_0$:

$$\begin{tikzcd}
\mathcal{F}_1(g)\otimes\mathcal{F}_1(h) \arrow[r, "\theta_g\otimes\theta_h"] \arrow[d, "\mathcal{F}_2"] & \mathcal{F}'_1(g)\otimes\mathcal{F}'_1(h) \arrow[d, "\mathcal{F}'_2"] \\
\mathcal{F}_1(g\otimes h) \arrow[r, "\theta_{g\otimes h}"] & \mathcal{F}'_1(g\otimes h)
\end{tikzcd}
\qquad
\begin{tikzcd}
1' \arrow[d, "\mathcal{F}_0"'] \arrow[dr, "\mathcal{F}'_0"] & \\
\mathcal{F}_1(1) \arrow[r, "\theta_1"] & \mathcal{F}'_1(1)
\end{tikzcd}$$

**Remark 5.** An isomorphism $\mathcal{F}_{-1_g} : \overline{\mathcal{F}(g)} \to \mathcal{F}(\overline{g})$ is uniquely determined by the commutativity of the following diagrams:
Compatibility with the unit and the counit:

$$\begin{tikzcd}
\mathcal{F}_1(g)\otimes\overline{\mathcal{F}_1(g)} \arrow[r, "1\otimes\mathcal{F}_{-1}"] & \mathcal{F}_1(g)\otimes\mathcal{F}_1(\overline{g}) \arrow[r, "\mathcal{F}_2"] & \mathcal{F}(g\otimes\overline{g}) \\
1' \arrow[u, "i_{\mathcal{F}(g)}"] \arrow[rr, "\mathcal{F}_0"] & & \mathcal{F}_1(1) \arrow[u, "\mathcal{F}(i_g)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\mathcal{F}_1(g)}\otimes\mathcal{F}_1(g) \arrow[r, "\mathcal{F}_{-1}\otimes 1"] \arrow[d, "e_{\mathcal{F}(g)}"] & \mathcal{F}(\overline{g})\otimes\mathcal{F}_1(g) \arrow[r, "\mathcal{F}_2"] & \mathcal{F}_1(\overline{g}\otimes g) \arrow[d, "\mathcal{F}(e_g)"] \\
1' \arrow[rr, "\mathcal{F}_0"] & & \mathcal{F}_1(1)
\end{tikzcd}$$

# 4 No strict 3-groupoid models the 2-sphere – Daniel Brügmann

As usual (his words), Peter jumps in to contextualize. Recall that Malte told us about 2-groupoids – almost. Really they are 2-categories with invertible morphisms, of course; Malte restricted to the case of a single object. Thus we have the inclusions $\texttt{2-GR} = \texttt{2-GRP}_0 \subset \texttt{2-GRP} \subset \texttt{2-CAT}$. If we replace 2 with 1, we just get $\texttt{GR} = \texttt{1-GRP}_0 \subset \texttt{1-GRP}$. We understand these well, so Malte's talk is the first nontrivial case. Now, recall that we have the classifying space functor $B : \texttt{2-GRP} \to \texttt{2-TYP} \subset \texttt{TOP}$, which restricts to $B : \texttt{2-GRP} \to \texttt{2-TYP}_0$.

But actually, not quite. The one unnatural thing in the previous talk is that we added a basepoint; from this, we concluded that we had a bijection between equivalence classes of objects of $\texttt{2-GRP}_{0,*}$ and $\texttt{2-TYP}_{0,*}$. But of course, these are really all *3-categories* – ideally, we should enrich our statement to say that we have an equivalence $\texttt{2-GRP} \simeq \texttt{2-TYP}$ (of $(3,1)$-categories). Peter challenges Dave's comment last week, and claims that this maps to a Quillen equivalence $\texttt{2-CAT} \xrightarrow{\sim} \texttt{BiCAT}$. Dave has quibbles. But Peter wants to progress downwards to $n = 1$.

So, here is the thing about baspoints: we only get 2-categories. This is reflected in topology, by the fact that if $X, Y \in k\text{-}\texttt{TYP}_{0,*}$ then $\pi_n(\mathrm{Map}_*(X,Y), \mathrm{const}) \cong [S^n \wedge X, Y]_*$, and this is trivial for $n \geq k$ since the source $S^n \wedge X$ only has $(n+1)$-cells and higher. Thus, $\mathrm{Map}_*(X,Y) \in (k-1)\text{-}\texttt{TYP}$. Thus, we can consider $k\text{-}\texttt{TYP}^{\times}_{0,*} \in k\text{-}\texttt{TYP}$ (where the $\times$ denotes that we are only taking homotopy equivalences, so that all our morphisms are invertible). Now, at $n = 1$, we have $B : \texttt{1-GRP} \to \texttt{1-TYP}$, which restricts to $B : \texttt{1-GRP}_0 \to \texttt{1-TYP}_0$. But in fact, we have a 2-category at $\texttt{1-GRP}_{0,*}$; 2-morphisms are given by multiplication by an element in the target. In fact, morally we should have that $\pi_2(\texttt{GR}^{\times}; G) = \pi_1(\texttt{GR}^{\times}(G,G), \mathrm{id}_G) = Z(G)$. (Of course, this is also $\pi_1(\mathrm{Map}(BG, BG), \mathrm{id}_{BG})$, the unpointed maps.) Thus, adding a basepoint kills the last categorical layer.

Now, we would like this analogy to continue upwards. Daniel will now shatter our dreams.

## 4.1 Outline

More precisely, we will see that no strict 3-groupoid models the 3-type of the 2-sphere.

Recall that we have a realization functor $R : n\texttt{CAT} \to \texttt{sSET}$, which restricts to $n\texttt{GROUPOID} \to n\text{-}\texttt{TYPES}$. The *homotopy hypothesis* asserts that this should always be an equivalence. What we will show is that this is *not* satisfied if we take *strict* 3-groupoids. We will write $S^2$ to mean its 3-type, and we will often drop the word "strict" too.

There are two ingredients to this proof.

1. First, we will define the homotopy groups of an $n$-groupoid (which will vanish above level $n$), and we will show that this commutes with realization: $\pi_i(\mathcal{A}) \cong \pi_i(R\mathcal{A})$ (naturally).
2. Then, we will do something on the 3-groupoid side that we can't do with 3-types. Namely, we have the following resut: if $\mathcal{C}$ is a 3-groupoid with $\pi_0(\mathcal{C}) = \pi_1(\mathcal{C}) = *$ and $\pi_2(\mathcal{C}) \cong \mathbb{Z}$, then there is a diagram of 3-groupoids

$$\begin{tikzcd}\mathcal{C} & \mathcal{A} \arrow[l, "\pi_*\text{-iso.}"'] \arrow[r, "\pi_3\text{-iso.}"] & \mathcal{D}\end{tikzcd}$$

where $\mathcal{D}$ is 2-connected.

This implies the main result as follows. Suppose we have a strict 3-groupoid $\mathcal{C}$ with $R\mathcal{C} \simeq S^2$. Then we get

$$\begin{tikzcd}R\mathcal{C} & R\mathcal{A} \arrow[l, "\pi_*\text{-iso.}"'] \arrow[r, "\pi_3\text{-iso.}"] & R\mathcal{D}\end{tikzcd}$$

by naturality. By naturality of the Hurewicz map, we get

$$\begin{tikzcd}
\pi_3(R\mathcal{A}) \arrow[r, "\cong"] \arrow[d] & \pi_3(R\mathcal{D}) \arrow[d, "\cong"] \\
H_3(R\mathcal{A}) \arrow[r] & H_3(R\mathcal{D}).
\end{tikzcd}$$

But of course $H_3(R\mathcal{A}) = H_3(S^2) = 0$ (since we can make the Postnikov truncation $p_3S^2$ by attaching only 5-cells and higher), so this becomes a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cong"] \arrow[d] & \mathbb{Z} \arrow[d, "\cong"] \\
0 \arrow[r] & \mathbb{Z},
\end{tikzcd}$$

which is impossible.

Simpson explains this as follows: the Whitehead pairing constructs the Hopf map, giving the nontrivial element of $\pi_3(S^2)$. But Whitehead pairings must vanish on any $R\mathcal{C}$.

Peter explains this as follows: the map $\mathcal{A} \to \mathcal{D}$ is a nontrivial map to a $K(\mathbb{Z}, 3)$, but this doesn't exist in spaces – $H^3(S^2; \mathbb{Z}) = 0$. This gets at the fact that strict 3-groupoids are *too strict*: there are too many maps between them.

## 4.2 The homotopy groups of groupoids

**Definition 17.** A *0-groupoid* is a set, and an *equivalence* of 0-groupoids is a bijection. Then, an *$n$-groupoid* $\mathcal{A}$ is an $n$-category such that:

- for all $x, y \in \mathrm{Ob}(\mathcal{A})$, $\mathcal{A}(x, y)$ is an $(n-1)$-groupoid;
- for all 1-morphisms $u \in \mathcal{A}(x, y)$ and for any $z \in \mathrm{Ob}(\mathcal{A})$, precomposition and postcomposition give equivalences of $(n-1)$-groupoids $u- : \mathcal{A}(y, z) \to \mathcal{A}(x, z)$ and $-u : \mathcal{A}(z, x) \to \mathcal{A}(z, y)$.

An *equivalence* of $n$-groupoids is a functor $F : \mathcal{A} \to \mathcal{B}$ of $n$-categories which is fully faithful (which is defined inductively) and essentially surjective (which can be defined simply as the existence of a morphism in either direction between any object and an object in the image of $F$, since our morphisms will be appropriately invertible). (For $X, Y \in \mathrm{Ob}(\mathcal{A})$, we say that $X$ is *equivalent* to $Y$, and write $X \sim Y$, iff there is a morphism $X \to Y$. This is obviously transitive and reflexive; to show symmetry, we use the definition of $n$-groupoid.)

We point out that the second condition on an $n$-groupoid is not as strict as one might imagine it should be. One might demand a more rigid invertibility condition. Our arguments go through in those cases too. Also, our composition goes the correct way, which is opposite from usual; this will make our pictures look better.

Now, we first define the homotopy groups as sets.

**Definition 18.** First, we define $\pi_0(\mathcal{A}) = \mathrm{Ob}(\mathcal{A})/\sim$. (This is obviously functorial. Note also that if $F : \mathcal{A} \to \mathcal{B}$ is a functor of $n$-groupoids, then $\pi_0(F)$ is surjective iff $F$ is essentially surjective.) Then, given $a \in \mathrm{Ob}(\mathcal{A})$ and $1 \leq i \leq n$, we define $\pi_i(\mathcal{A}, a) = \pi_{i-1}(\mathcal{A}(a, a), 1_a)$. (This is functorial by induction.)

We now define our functor $R : n\texttt{GROUPOIDS} \to \texttt{sSET}$. Now, we can consider $n\texttt{GROUPOIDS} \subset ((n-1)\texttt{GROUPOIDS})\texttt{-CAT}$ (i.e. categories enriched in $(n-1)$-groupoids), and we already know how to realize the latter, as $((n-1)\texttt{GROUPOIDS})\texttt{-CAT} \to \texttt{sSET-CAT}$. From here, we realize to bisimplicial sets as $\texttt{sSET-CAT} \subset \texttt{sCAT} \xrightarrow{N_*} \texttt{ssSET}$ (i.e. the composition $\Delta^{op} \to \texttt{CAT} \to \texttt{sSET}$). Then, we take realization (i.e. precomposition with the diagonal) to get to $\texttt{TOP}$. Thus, the entire diagram is

$$R : n\texttt{GROUPOIDS} \subset ((n-1)\texttt{GROUPOIDS})\texttt{-CAT} \xrightarrow{\text{induction}} \texttt{sSET-CAT} \subset \texttt{sCAT} \xrightarrow{N_*} \texttt{ssSET} \xrightarrow{\Delta^*} \texttt{sSET}.$$

(Typically one uses the *homotopy-coherent* nerve, but this is actually hidden in our argument.)

**Proposition 5.** *$\pi_i(\mathcal{C}, c) \cong \pi_i(R\mathcal{C}, c)$.*

*Proof sketch.* We use the following fact: if $\mathcal{C} \in \texttt{sCAT}$, then $h\mathcal{C}$ is a groupoid with $\mathrm{Ob}(h\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ and $h\mathcal{C}(X,Y) = \pi_0(\mathcal{C}(X,Y))$. This implies that $\pi_i(|N_*\mathcal{C}|, c) \cong \pi_{i-1}(\mathcal{C}(c,c), \mathrm{id}_c)$. (This is like taking loopspaces.)

Now, our proof will go by induction on the previous $R$, say $R_{n-1} : (n-1)\texttt{GROUPOIDS} \to \texttt{sCAT}$. Then,

$$\pi_i(|N_*R\mathcal{A}|, a) \cong \pi_{i-1}((R\mathcal{A})(a,a), 1_a) = \pi_{i-1}(R(\mathcal{A}(a,a)), 1_a) \cong \pi_{i-1}(\mathcal{A}(a,a), 1_a) = \pi_i(\mathcal{A}, a).$$

□

Now, we give the group structure on the homotopy groups of $n$-groupoids. First, if $\mathcal{A} \in n\texttt{GROUPOID} \subset n\texttt{-CAT}$, then $\mathcal{A}(a,a)$ is a monoid-object in $(n-1)\texttt{-CAT}$. Then, $\pi_1(\mathcal{A}, a) = \pi_0(\mathcal{A}(a,a), 1_a) = \mathrm{Ob}(\mathcal{A}(a,a))/\sim$. But composition is invertible up to equivalence, so this is not just a monoid but a group. Then of course, we define $\pi_i$ for $i \geq 2$ by induction. We can use the Eckmann-Hilton argument to see that $\pi_2(\mathcal{A}, a)$ is an abelian group. Namely, $\pi_2(\mathcal{A}, a) = \pi_1(\mathcal{A}(a,a), 1_a)$, but since $\mathcal{A}(a,a)$ is an $(n-1)$-category, then $(\mathcal{A}(a,a))(1_a, 1_a)$ is a monoid-object in $(n-2)\texttt{-CAT}$. But now Eckmann-Hilton says that abelian-monoid-obects in $(n-2)\texttt{-CAT}$ are the same thing as monoid-objects in the category of monoid-objects in $(n-2)\texttt{-CAT}$ (which is a non-full subcategory – the only morphisms allowed are those that preserve the monoid structure). Note that there are two compositions on the latter: we will denote the horizontal composition by $+$.

### 4.3 The auxiliary result

For $n \geq 2$ we have an equivalence of 1-categories between one-object one-1-morphism $n$-categories and abelian-monoid-objects in $(n-2)\texttt{-CAT}$; this takes $\mathcal{C}$ to $(\mathcal{C}(c,c))(1_c, 1_c)$. This will allow us to prove our result.

*Proof sketch.* There will actually be an intermediate 3-groupoid $\mathcal{B}$ between $\mathcal{C}$ and $\mathcal{A}$. Namely, we define $\mathcal{B}$ to have one object $c$, with $\mathcal{B}(c,c) = \{1_c\}$ and with $(\mathcal{B}(c,c))(1_c, 1_c) = (\mathcal{C}(c,c))(1_c, 1_c)$. Then we have $\mathcal{C} \hookleftarrow \mathcal{B}$, which is a $\pi_*$-isomorphism.

The construction of $\mathcal{A}$ is more complicated, and in fact it will be easier to first define $\mathcal{D}$ (depending on $\mathcal{C}$). We set $\mathrm{Ob}(\mathcal{D}) = \{c\}$, with one 1-morphism and one 2-morphism, and with $((\mathcal{D}(c,c))(1_c, 1_c))(1_{1_c}, 1_{1_c}) = ((\mathcal{C}(c,c))(1_c, 1_c))(1_{1_c}, 1_{1_c})$. Note that this agrees with the 3-morphisms in $\mathcal{B}$.

Finally, we get $\mathcal{A}$ by changing the set of objects. In the equivalence of 1-categories given at the beginning of this subsection, we observe that an $n$-groupoid gives us an abelian-monoid-object in $(n-2)\texttt{GROUPOIDS} \subset (n-2)\texttt{-CAT}$; moreover, this assignment is *fully faithful* (in a sense we won't explicitly define). (This is a statement about compatibility with the abelian-monoid-object structure.) We say this because we will construct the map $\mathcal{A} \to \mathcal{D}$ in abelian-monoid-objects in $(n-2)\texttt{GROUPOIDS}$. We start at the adjunction $\mathrm{Ob} : 1\texttt{GROUPOIDS} \leftrightarrows \texttt{SET} : \text{codiscrete}$ (where "codiscrete" means we have exactly one morphism between any two objects). Now, $\mathcal{B}$ gives us an abelian-monoid-object $\mathcal{G}$ in $(n-2)\texttt{GROUPOIDS}$, and we define $\mathcal{A}$ to give us

$$\mathcal{G}' = \text{codiscrete}(\mathbb{N} \times \mathbb{N}) \times_{\text{codiscrete}(\mathrm{Ob}(\mathcal{G}))} \mathcal{G}.$$

Specifically, the map $\mathbb{N} \times \mathbb{N} \to \mathrm{Ob}(\mathcal{G})$ is given by choosing a representative $a$ of $1 \in \mathbb{Z} \cong \pi_0(\mathcal{G}) = \pi_2(\mathcal{B}) \cong \pi_2(\mathcal{C})$, and similarly a representative $b$ of $-1$. Then we declare that $(1,0) \mapsto a$ and $(0,1) \mapsto b$ (and extend by the monoid structure; note that the codiscrete functor preserves monoid structure since it's a right adjoint).

Now, $\mathcal{D}$ corresponds to some abelian-monoid-object $\mathcal{H}$ in $(n-2)$`GROUPOIDS`, where $\mathrm{Ob}(\mathcal{H}) = \{1_{1_c}\}$ and $\mathcal{H}(1_{1_c}, 1_{1_c}) = \mathcal{G}(1_{1_c}, 1_{1_c})$.

So to finish, we have the following key fact. Suppose we have a morphism $\varphi : (0,0) \to (1,1)$ in $\mathcal{G}'$ (note that $\mathrm{Ob}(\mathcal{G}') = \mathbb{N} \times \mathbb{N}$); this exists because these are both sent to $0 \in \pi_0(\mathcal{G})$. Then, we claim that for any $k \in \mathbb{Z}$, every morphism $\alpha : (m,n) \to (m+k, n+k)$ can be uniquely written as $\alpha = 1_{m,n} + k \cdot \varphi + u$, where $u \in \mathcal{G}'((0,0),(0,0))$. (We denote by $\varphi$ its translates, too.) (This proof is rather involved, and uses an interchange law that we won't get into.) Then, our functor $\mathcal{A} \to \mathcal{D}$ is defined via the functor $\mathcal{G}' \to \mathcal{H}$ given by $\alpha \mapsto u \in \mathcal{H}(0,0)$. This completes the proof. $\square$

There is a question about the existence of $\mathcal{A}$ as defined via $\mathcal{G}'$, since the functor from $n$-groupoids to abelian-monoid-objects in $(n-2)$-groupoids may not be essentially surjective. But in fact, this can be checked by hand.

Chris adds: It might initially seem confusing that we need to go through $\mathcal{A}$ from $\mathcal{B}$ to $\mathcal{D}$. But he can give us an explicit example where we need it. Note that $\mathcal{D}$ is a $K(\mathbb{Z}, 3)$. Then, suppose we have that $\mathcal{B}$ is 1-connected and has 2-morphisms given by $\mathbb{Z}/2 \times \mathbb{Z}$, and with 3-morphisms $(\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z})$ with the source and target maps $(\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}) \rightrightarrows \mathbb{Z}/2 \times \mathbb{Z}$ given by $s(x,m,n) = (x,n)$ and $t(x+m,n)$. So as a category, this splits as a lone copy of $\mathbb{Z}$ and a copy of $(\mathbb{Z}/2 \times \mathbb{Z} \rightrightarrows \mathbb{Z}/2 \times \mathbb{Z})$. Now, the automorphisms of the second factor are exactly the elements that get sent to the identity, i.e. $2\mathbb{Z} \subset \mathbb{Z}$. But this doesn't have a $\pi_3$-isomorphism to $\mathcal{D}$; at best, we can hit $2\mathbb{Z} \subset \mathbb{Z}$.

## 5 Complete Segal spaces and Segal categories – Alexander Körschgen

Peter announces that the seminar schedule has changed: we will be bypassing the $E_n$ stuff, and we will be going directly to infinity and beyond! Today we will see that weak $n$-groupoids are equivalent to $n$-types.

A quick note. Suppose $\mathcal{C}$ is a (small) category, and we have objects $X_i \in \mathcal{C}$. Then of course we know what the nerve $N(\mathcal{C})$ is supposed to be. But we spell this out in detail, since not everyone may have thought through it. Of course vertices are objects and morphisms are edges. Then, triangles witness compositions $\varphi_{12} \circ \varphi_{01} = \varphi_{02}$ (for morphisms $\varphi_{ij} : X_i \to X_j$). Then, associativity translates to a *well-defined* choice for labeling the interior of a 3-simplex, namely $\varphi_{13} \circ \varphi_{01} = \varphi_{23} \circ \varphi_{02}$. This should give some idea of why higher category theory can be embedded into the theory of simplicial sets – "internal simplicial sets" should be thought of as a generalization of "internal categories".

And now, on to Alexander's talk!

### 5.1 Definitions

We will study Segal categories. We will think of the nerve as a set of vertices $NC_0$, and sets $NC(a_0, \ldots, a_k)$ for each $(a_0, \ldots, a_k) \in (NC_0)^{k+1}$; we consider this set as $\{(f_1, \ldots, f_k) : sf_1 = a_0, tf_1 = sf_2 = a_1, \ldots\}$. This satisfies $NC(a_0, \ldots, a_k) \cong NC(a_0, a_1) \times NC(a_1, a_2) \times \cdots \times NC(a_{k-1}, a_k)$.

With this in hand, we give the following definition.

**Definition 19.** Let $A$ be a category with products. Then an *$A$-simplicial set* $S$ consists of a set $S_0$ and objects $S(a_0, \ldots, a_k) \in A$ for each $(a_0, \ldots, a_k) \in (S_0)^{k+1}$ together with maps $\sigma^* : SS(a_0, \ldots, a_k) \to S(a_{\sigma(0)}, \ldots, a_{\sigma(k)}) \in A$ for each $\sigma : [k] \to [m] \in \Delta$, such that:

- $(\sigma_1 \circ \sigma_2)^* = \sigma_2^* \circ \sigma_1^*$ (and these satisfy the simplicial identities);
- $S(a) \in A$ is terminal in $A$.

**Remark 6.** We view the maps $[n] \to [0] \in \Delta$ as giving us $S(a) \to S(a, \ldots, a)$ for all $a \in A$.

**Remark 7.** This is similar to the idea of an internal category (i.e. a simplicial object), although note that $S$ really is a *set*. Of course, there will be cases that we can consider sets as living in $A$.

We want to define the categories $(\infty, n)$-`Cat` and $n$-$\texttt{Cat}^W$ recursively. These are equipped with a notion of equivalence, as well has a ("homotopy category") functor $h^n$ to `Cat` (usual strict categories) that preserves products and equivalences, and a functor $\pi_0^n$ to `Sets` which preserves products and sends equivalences to isomorphisms of sets. (Of course, $\pi_0^n = \pi_0 \circ h^n$.)

**Definition 20.**

- At $n = 0$, we have that a *weak 0-category* is a set; equivalences are bijections, $h^0$ is the usual discrete embedding, and so $\pi_0^0$ is the identity functor.
- Meanwhile, an $(\infty, 0)$*-category* is a space that is homotopy equivalent to a CW-complex; equivalences are homotopy equivalences, $h^0$ is the fundamental groupoid functor, and so $\pi_0^0$ is the usual $\pi_0$ functor.
- A *weak n-category* or $(\infty, n)$*-category* is an enriched simplicial set $C$ in [the same thing but with $n-1$] such that for all tuples $(a_0, \ldots, a_k) \in (C_0)^{k+1}$ with $k \geq 2$, the map

$$C(a_0, \ldots, a_k) \to C(a_0, a_1) \times \cdots \times C(a_{k-1}, a_k)$$

induced by the edge inclusions $(i, i+1) \to [k]$ is an equivalence of [$n-1$ versions].

This last condition is called the *Segal condition.*

Note that this is already weaker than our previous conditions at $(\infty, 1)$-categories: we have "spaces of $k$-morphisms", and this is only required to be *homotopy equivalent* to the evident composition, whereas it used to have to be an *isomorphism.* We will see in a future talk that these are Quillen equivalent to, but not the same as, quasicategories.

**Definition 21.** An $(\infty, n)$-category is also called a *Segal n-category.*

**Definition 22.** Let $X$ be a weak $n$-category or an $(\infty, n)$-category. Then we define the *homotopy category* $h^n(X) \in \texttt{Cat}$ to have objects $X_0$ and morphisms determined inductively by $(h^n C)(x, y) = \pi_0^{n-1}(C(x, y))$. The composition

$$(h^n C)(x, y) \times (h^n C)(y, z) \to h^n(x, z)$$

is defined by

$$(h^n C)(x, y) \times (h^n C)(y, z) = \pi_0^{n-1}(C(x, y) \times C(y, z)) \cong \pi_0^{n-1} C(x, y, z) \xrightarrow{h} \pi_0^{n-1}(C(x, z)) = h^n(x, z).$$

The point here is that the Segal condition is only a homotopy equivalence, but it induces an isomorphism on $\pi_0$.

**Definition 23.** We define the functor $\pi_0^n$ by $\pi_0^n C = \pi_0 h^n C$ (i.e. the objects of $h^n C$ up to isomorphism).

**Definition 24.** A simplicial map $f : C \to D$ between weak $n$-categories or $(\infty, n)$-categories is an *equivalence* if:

- $C(a, b) \to D(f(a), f(b))$ is an equivalence of [$n-1$ guys] for all $a, b \in C_0$, and;
- $h^n C \to h^n D$ is an equivalence of categories.

This second condition is slightly stronger than we need, but it will suffice for our purposes.

**Remark 8.** We have the inclusion `Sets` $\to$ `Top`, and this induces a fully faithful functor $n$-$\texttt{Cat}^W \to (\infty, n)$-`Cat`. (And of course, we have $n$-`Cat` $\to n$-$\texttt{Cat}^W$.) So we will mostly work in the latter context, but most of the things we do will go through for weak $n$-categories.

**Remark 9.** A weak 1-category is just a strict small 1-category. However, the notion of equivalences is weaker.

## 5.2 Groupoids

**Definition 25.** We declare that every $(\infty, 0)$-category is an $(\infty, 0)$-*groupoid*. Then, an $(\infty, n)$-category is an $(\infty, n)$-*groupoid* if all $C(a_0, \dots, a_k)$ are $(\infty, n-1)$-groupoids and $h^nC$ is a groupoid in the classical sense.

**Definition 26.** For an $(\infty, 0)$-groupoid, the *homotopy groups* $\pi_i^0$ are the classical homotopy groups of the space. If $C$ is an $(\infty, n)$-groupoid, we define its *homotopy groups* based at any $c \in C_0$ by $\pi_i^n(C, c) = \pi_{i-1}^{n-1}(C(c, c), \mathrm{id}_c)$. (Note that we obtain $\mathrm{id}_c$ as the map from the terminal object of $C$ to $C(c, c)$.)

We want to define realization functors to spaces which commute with taking homotopy groups. We denote these by $B^n : (\infty, n)\text{-}\mathtt{Cat} \to \mathtt{Top}$.

**Definition 27.** At $n = 0$ we just define the *realization functor* $B^0$ to be the identity. Then if $C$ is an $(\infty, n)$-category, we define a simplicial space $sC$ by taking $sC_0 = C_0$ and $sC_k = \coprod_{(a_0,\dots,a_k) \in sC_0^{k+1}} B^{n-1}C(a_0, \dots, a_k)$. Then we define $B^n$ to be the composition of this with the realization functor.

**Theorem 2** (Segal, Tamsamani)**.** *If $C$ is an $(\infty, 1)$-groupoid, then we have a map $C(x, y) \times \Delta^1 \to B^1C$, and this admits an adjoint $C(x, y) \to P_{x,y}(B^1C)$ as a map into the path space from $x$ to $y$. This is a weak equivalence for all $x, y \in C_0$.*

**Corollary 2.** *By induction, it follows that if $C$ is an $(\infty, n)$-groupoid, then $B^{n-1}C(x, y) \xrightarrow{\sim} P_{x,y}B^nC$.*

This will allow us to show that our notion of homotopy groups really do make sense, and are compatible with realization.

We need the following result to prove the theorem.

**Lemma 3.** *If $C$ is an $(\infty, n)$-groupoid, then $\pi_i(B^nC, c) \cong \pi_i^n(C, c)$.*

*Proof.* We go by induction. At $n = 0$ this is a definition. For $n > 0$, we have $\pi_0(B^nC)$ is given by $C_0$ up to paths generated by the $B^{n-1}C(x, y)$. (A quick way to see this is that $\pi_0 : \mathtt{Top} \to \mathtt{Sets}$ is a left adjoint, so we can compute $\pi_0$ of a simplicial space by first applying $\pi_0$ levelwise and then computing the colimit in $\mathtt{Sets}$.) On the other hand, $\pi_0^n(C, c) = \pi_0 h^nC$, which is $C_0$ up to isomorphisms, but since we're in a groupoid then this is just $C_0$ up to morphisms. So, we get $\pi_0(|C(x, y)|)$.

Then, for all $i > 0$ we have a map $B^{n-1}C(c, c) \to P_{c,c}B^nC$, and this is an equivalence. Therefore,

$$\pi_i^n(C, c) = \pi_{i-1}^{n-1}(C(c, c), \mathrm{id}_c) = \pi_{i-1}(B^{n-1}C(c, c), \mathrm{id}_c) \cong \pi_{i-1}(\Omega_c B^nC, \mathrm{const}_c) = \pi_i(B^nC, c).$$

□

One could now prove the theorem. We won't. Rather, we would like to see why realization gives an equivalence onto $n$-`Types` (i.e. fixes the problems we had with strict $n$-categories).

## 5.3 Relation with $n$-`Types`

We construct functors $\Pi^n : (\infty, n)\text{-}\mathtt{Grpds} \to (\infty, n+1)\text{-}\mathtt{Grpds}$ which preserve products and commute with taking the homotopy category.

So at $n = 0$, let $X$ be a space. Then we set $(\Pi^0X)_0 = X_{\text{discrete}}$, and $(\Pi^0X)(a_0, \dots, a_k) = \mathrm{map}(\Delta^k, X)_{a_0,\dots,a_k}$. This satisfies the Segal condition because

$$\mathrm{map}(\Delta^k, X)_{a_0,\dots,a_k} \to \mathrm{map}(\Delta^1, X)_{a_0,a_1} \times \cdots \times \mathrm{map}(\Delta^1, X)_{a_{k-1},a_k}$$

is induced by the *spine inclusion* (the longest ordered string of edges in the $k$-simplex), which is a homotopy equivalence and which therefore induces a homotopy equivalence of mapping spaces. It is clear that this preserves products since we're mapping into our spaces. Lastly, we have that $(h^1(\Pi^0 X))(x,y) = \pi_0(\mathrm{map}(\Delta^1, X)_{x,y}) = (\pi_{\leq 1}X)(x,y) = (h^0 X)(x,y)$. So the homotopy category is preserved: $h^1 \circ \Pi^0 = h^0$.

Now for $n > 0$, let $X$ be an $(\infty, n)$-groupoid. Again we set $(\Pi^n X)_0 = X_0$. Now, we define $(\Pi^n X)(a_0, \ldots, a_k) = \Pi^{n-1}(X(a_0, \ldots, a_k))$. One can show by induction that this preserves the homotopy category as well, satisfies the Segal condition, and so on.

More importantly, though, this construction comes with a map $B^{n+1}\Pi^n X \to B^n X$. At $n = 0$, this is $B^1\Pi^0 X \to B^0 X$, and by the adjunction this is equivalent to a map $s\Pi^0 X \to \mathrm{map}(\Delta^\bullet, X)$. For $n > 0$, we define the map inductively. Then, the crucial result is the following.

**Proposition 6.** *The map $B^{n+1}\Pi^n X \to B^n X$ is a weak equivalence.*

Given an $n$-type $X$, we can send it through the functors

$$(\infty, 0)\text{-}\mathtt{Grpd} \xrightarrow{\Pi^0} \cdots \xrightarrow{\Pi^{n-1}} (\infty, n)\text{-}\mathtt{Grpds},$$

with $B^n(\Pi^{n-1} \circ \cdots \circ \Pi^0 X) \simeq X$. (Actually, this is true for any space.) On the other hand, we can define a *discretization functor* $\widetilde{\pi}_0^n : (\infty, n)\text{-}\mathtt{Grpds} \to n\text{-}\mathtt{Grpd}^W$, which is defined inductively by taking the usual $\pi_0 : \mathtt{Top} \to \mathtt{Sets}$. This comes with a natural transformation $\mathrm{Id} \to \widetilde{\pi}_0^n$ (i.e. $\widetilde{\pi}_0^n$ followed by the inclusion), which has that

$$\pi_i(\widetilde{\pi}_0^n X) = \begin{cases} \pi_i(X, x), & i \leq n \\ 0, & i > n. \end{cases}$$

So this kills all homotopy groups above level $n$. Therefore, $\widetilde{\pi}_0^n(\Pi^{n-1} \circ \cdots \circ \Pi^0 X)$ has all the same homotopy groups as $X$ as well.

So to summarize, we have the following theorem.

**Theorem 3.** *If $X$ is an $n$-type, there is a weak $n$-groupoid $C$ such that $B^n CC \simeq X$.*

This passes through

$$(\infty, 0)\text{-}\mathtt{Grpd} \xrightarrow{\Pi^{n-1} \circ \cdots \circ \Pi^0} (\infty, n)\text{-}\mathtt{Grpd} \xrightarrow{\widetilde{\pi}_0^n} n\text{-}\mathtt{Grpd}^W,$$

under which if we have $X \mapsto \hat{X} \mapsto C$ then $B^n C \simeq B^n \hat{X} \simeq X$.

### 5.4 Examples

Peter collects examples from the audience.

**Example 15.** Given an $n$-type, we get a weak $n$-groupoid. This is already way better than strict $n$-groupoids.

**Example 16.** We will construct a functor *catBiCat* $\to 2\text{-}\mathtt{Cat}^W$ next week, called the *coherent nerve.* If we actually start with a strict 2-category, then we can get two different weak 2-groupoids: either we can take the nerve, or we can take the coherent nerve. It will turn out that the latter is *better*, since it will automatically be fibrant.

## 6 Examples and problems with weak $n$-categories – Kim Nguyen

### 6.1 Background

We begin by recalling that an *$n$-category* $\mathcal{C}$ is a simplicial set enriched in $(n-1)$-categories such that the Segal maps

$$\mathcal{C}(a_0, \ldots, a_k) \xrightarrow{\sim} \mathcal{C}(a_0, a_1) \times \cdots \times \mathcal{C}(a_{k-1}, a_k)$$

are equivalences of $(n-1)$-categories for all tuples $(a_0, \ldots, a_k) \in (\mathcal{C}_0)^{k+1}$. (A 0-category is just a set.)

**Example 17.** Let $\mathcal{C}$ be a strict $n$-category. Taking the nerve $N\mathcal{C}$ gives a simplicial object in strict $(n-1)$-categories. Moreover, $N\mathcal{C}_0$ is discrete, and in fact the Segal maps are isomorphisms. Taking fibers as

$$\begin{tikzcd}
N\mathcal{C}(a_0, \dots, a_k) \arrow[r] \arrow[d] & N\mathcal{C}_k \arrow[d] \\
(a_0, \dots, a_k) \arrow[r] & N\mathcal{C}_0 \times \cdots \times N\mathcal{C}_k
\end{tikzcd}$$

yields our weak $n$-category. (More precisely, we should give our functor from strict $n$-categories to weak $n$-categories inductively, so that we can consider a simplicial object in strict $(n-1)$-categories as a simplicial object in weak $(n-1)$-categories; of course, isomorphisms will be taken to equivalences.)

So now we have three different types of 2-dimensional categories, which sit in the (non-commutative) diagram

$$\begin{tikzcd}
\text{strict 2-categories} \arrow[r, shift left, "\text{inclusion}"] \arrow[dr, "\text{nerve}"'] & \text{bicategories} \arrow[l, shift left, "\text{strictify}"] \arrow[d, dashed, "\text{2-nerve}"] \\
 & \text{2-categories;}
\end{tikzcd}$$

today we will study the *2-nerve*.

## 6.2 The setup

We recall that the usual nerve is induced by the inclusion $\mathbf{\Delta} \hookrightarrow$ `Cat` $\hookrightarrow$ `Bicat`. This would give us the usual nerve. Instead, we will consider the strict 2-category `NHom`, whose objects are bicategories, whose 1-morphisms are bicategory homomorphism, and whose 2-morphisms are *icons*: "identity component oplax natural transformations".

Recall that a homomorphism $(F, \varphi) : \mathcal{D} \to \mathcal{D}'$ of bicategories consists of:

- a function $F : \mathrm{ob}(\mathcal{D}) \to \mathrm{ob}(\mathcal{D}')$;
- functors $F : \mathcal{D}(A, B) \to \mathcal{D}'(FA, FB)$ for all $A, B \in \mathrm{ob}(\mathcal{D})$;
- natural northeast-oriented isomorphisms $\varphi$ in the diagrams

$$\begin{tikzcd}
\mathcal{D}(B, C) \times \mathcal{D}(A, B) \arrow[r] \arrow[d] & \mathcal{D}'(FC, FB) \times \mathcal{D}'(FB, FA) \arrow[d] \\
\mathcal{D}(A, C) \arrow[r] & \mathcal{D}'(FA, FC);
\end{tikzcd}$$

- a natural northeast-oriented isomorphism $\varphi^0$ in the diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & \mathcal{D}(A, A) \arrow[d] \\
A \arrow[r] & \mathcal{D}'(FA, FA).
\end{tikzcd}$$

There are of course axioms for the associators and unitors, but we won't write them down. This homomorphism is called *normal* if $\varphi^0$ the identity.

**Definition 28.** Let $(F, \varphi), (G, \psi) : \mathcal{A} \to \mathcal{D}$ be two normal homomorphisms. An *icon* can only exist if $F(A) = G(A)$ for all $A \in \mathrm{ob}(\mathcal{A})$, and in this case consists of a natural downward-oriented transformation $\alpha$ in the diagram

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[r, "F"] \arrow[d, equal] & \mathcal{D}(FA,FB) \arrow[d, equal] \\
\mathcal{A}(A,B) \arrow[r, "G"'] & \mathcal{D}(FA,FB),
\end{tikzcd}$$

such that the vertical composition

$$\begin{tikzcd}
\mathcal{A}_2 \arrow[r, shift left, "F\times F"] \arrow[r, shift right, "G\times G"'] \arrow[r, phantom, "\alpha \times \alpha"] \arrow[d] \arrow[dr, phantom, "\psi"] & \mathcal{D}_2 \arrow[d] \\
\mathcal{A}_1 \arrow[r, "G"'] & \mathcal{D}_1
\end{tikzcd}$$

is equal to the vertical composition

$$\begin{tikzcd}
\mathcal{A}_2 \arrow[r, "F\times F"] \arrow[d] \arrow[dr, phantom, "\varphi"] & \mathcal{D}_2 \arrow[d] \\
\mathcal{A}_1 \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] \arrow[r, phantom, "\alpha"] & \mathcal{D}_1.
\end{tikzcd}$$

(One can also write that "$\alpha\mathrm{Id} = \mathrm{Id}$".)

**Proposition 7.** `NHom` *is a strict 2-category.*

*Proof.* See Stephen Lack's paper on icons. ☐

### 6.3 The 2-nerve

Now, we will define an inclusion $\mathbf{\Delta} \hookrightarrow$ `NHom`, which will induce the 2-nerve, as follows.

First, there is a high-minded perspective. Recall that in the diagram

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[r] \arrow[d] & \mathtt{sSet} \\
\mathtt{Cat}, &
\end{tikzcd}$$

since every simplicial set is the colimit of its simplices, then we have a unique extension `Cat` ← `sSet`. This admits a right adjoint `Cat` → `sSet`; this is the ordinary nerve functor. On the other hand, in this situation we have

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[r] \arrow[d] & \texttt{2-Cat} \\
\texttt{NHom},
\end{tikzcd}$$

and this time the right adjoint to the left Kan extension will be the *2-nerve.*

But let us give a hands-on description as well. Consider $\texttt{NHom}([n], \mathcal{D})$ for some bicategory $\mathcal{D}$. Then a normal homomorphism consists of the data of objects $B_i \in \mathrm{ob}(\mathcal{D})$ for $0 \leq i \leq n$, along with 1-morphisms $\beta_{ij} : B_i \to B_j$ whenever $i < j$, and an downward-oriented invertible 2-cell $\beta_{ijk}$ in the diagram

$$\begin{tikzcd}
 & B_j \arrow[dr] & \\
B_i \arrow[ur] \arrow[rr] & & B_k
\end{tikzcd}$$

whenever $i < j < k$, such that the evident tetrahedron commutes, i.e. that the diagram of morphisms

$$\begin{tikzcd}
\beta_{kl} \circ \beta_{jk} \circ \beta_{ij} \arrow[r, "\beta_{jkl} \circ \mathrm{id}_{\beta_{ij}}"] \arrow[d, "\mathrm{id}_{\beta_{kl}} \circ \beta_{ijk}"'] & \beta_{jl} \circ \beta_{ij} \arrow[d, "\beta_{ijl}"] \\
\beta_{jk} \circ \beta_{ik} \arrow[r, "\beta_{ikl}"'] & \beta_{il}
\end{tikzcd}$$

commutes in $\mathcal{D}(B_i, B_l)$. (One generally considers the commutativity of an $n$-simplex by looking at the two canonical morphisms from 0 to $n$, namely the direct edge and the spine edge, and then demanding an equality of $(n-1)$-morphisms.) Then, an icon between two normal homomorphisms $(B, \beta) \to (C, \gamma)$ can only exist if $B_i = C_i$, and then consists of downward-oriented 2-cells $\varphi_{ij}$ in the diagrams

$$\begin{tikzcd}
B_i \arrow[r, "b_{ij}"] \arrow[d, equal] & B_j \arrow[d, equal] \\
B_i \arrow[r, "c_{ij}"'] & B_j,
\end{tikzcd}$$

such that the vertical composition of $\beta_{ijk}$ followed by $\varphi_{ik}$ followed by $\gamma_{ijk}^{-1}$ equals the horizontal composition of $\varphi_{ij}$ with $\varphi_{jk}$. (One should picture this as two triangles that share vertices, along with three bigons running between their corresponding edges.)

Now, given a bicategory $\mathcal{D}$, we will obtain $N\mathcal{D} \in [\mathbf{\Delta}^{\mathbf{op}}, \texttt{Cat}]$. Namely, for each $[n]$ we get $\texttt{NHom}([n], \mathcal{D})$ as above. (This uses the fact that $\texttt{NHom}$ is a strict 2-category.) This is the 2-nerve.

Let's look at the simplicial levels $n \geq 2$:

- $N\mathcal{D}_0$ is the discrete category whose objects are those of $\mathcal{D}$;
- $N\mathcal{D}_1$ has objects the 1-cells of $\mathcal{D}$ and morphisms the 2-cells;

- $N\mathcal{D}_2$ has objects the 2-commuting triangles and morphisms given by the "two triangles plus three bigons" situation above.

**Proposition 8.** *$N\mathcal{D}$ is a 2-category.*

*Proof.* We already saw that $N\mathcal{D}_0$ is discrete, so it remains to check the Segal maps. In fact, we claim that the Segal map

$$N\mathcal{D}_k \to N\mathcal{D}_1 \times_{N\mathcal{D}_0} \cdots \times_{N\mathcal{D}_0} N\mathcal{D}_1$$

is a surjective equivalence. For surjectivity, note that an object of the fiber product $(\mathcal{D}_1)^{\times_{N\mathcal{D}_0}k}$ is a string of composable 1-cells. We can hit this by taking the associated $(k+1)$-gon given by pasting together the triangles.

We will skip full-faithfulness, but the idea is quite similar. ☐

Note that for an $n$-simplex in the usual nerve, we do nothing more than choose objects for vertices and morphisms for edges, and then check things about the faces. In the 2-nerve, we choose objects for vertices, morphisms for edges, and 2-cells for faces, and then check things about the 3-cells. (This can be thought of some sort of coskeletal filtration towards the homotopy-coherent nerve.)

**Example 18.** Here is an example which will illustrate the advantage of the 2-nerve over the ordinary nerve. Note that we've talked about $n$-categories, but we haven't really talked about their morphisms. For example, let $G$ be a group and let $A$ be an abelian group. These are associated to 1-object 2-groupoids $\mathcal{G} = (G \Rrightarrow G \rightrightarrows \text{pt})$ and $\mathcal{A} = (A \Rrightarrow \text{pt} \rightrightarrows \text{pt})$. Now, taking nerve (i.e. considering simplicial maps) would give no nontrivial morphisms. But we want the homotopy hypothesis to hold, so this can't be right. On the other hand, $B^2\mathcal{G} \simeq BG$ and $B^2\mathcal{A} = K(A,2)$, so we should be getting $[BG, K(A,2)] = H^2(G,A)$.

So instead, let's take the 2-nerve of $\mathcal{A}$. This has $N\mathcal{A}_0 = (\text{pt} \rightrightarrows \text{pt})$ and $N\mathcal{A}_1 = (A \rightrightarrows \text{pt})$ as before, but then $N\mathcal{A}_2$ has objects $A$. It turns out that we will get $G^2 \to A$, and this will be nontrivial. (This defines *second* cohomology since we're in the normal case.)

The point here is that in the ordinary nerve, $\mathcal{G}$ only has objects the elements of $G$ along the edges and no morphisms, whereas $\mathcal{A}$ has only one object but interesting morphisms along the edges. So, there's no interaction between them where there should be. Again, the difference is that in the 2-nerve, the faces are *data* instead of *conditions* (which in turn get bumped up to the tetrahedra).

# 7 The "group-like" realization lemma – Dimitar Kodjabachev

Peter reminds us of what's going on today: we're proving a black box lemma, which we use for our inductive definition $|-| : (\infty, n)$-`groupoids` $\to$ `Top` (which preserved homotopy groups). It turns out that it's enough to prove this for $n = 1$. And in fact, we'll soon see that we can also model $(\infty,1)$-groupoids by Kan complexes, and this identification preserves geometric realization. And there's a beautiful combinatorial definition for the homotopy groups of a Kan complex, which allows us to easily compute the homotopy groups of the corresponding $(\infty,1)$-groupoid.

Before handing over the chalk to Dimitar, Peter shows us one cool step in the proof. Suppose $X$ is a (compactly generated) space; then we have $S_\bullet X$, its singular sset. It's rather obvious from the definitions that the combinatorial homotopy groups of $S_\bullet X$ coincide with the ordinary homotopy groups of $X$; then, the previous result to which we alluded implies that $|S_\bullet X|$ also has the same homotopy groups, and this ends up being a cofibrant replacement for $X$. On the other hand, we can instead define a simplicial *space* $S_\bullet X_{space}$ (with its compact-open topology and then the compactly-generated topology from there). Let's just refer to this as $S_\bullet X$, and refer to the previous one as $S_\bullet X_\delta$ ($\delta$ for "discrete"). Then, there is a map $S_\bullet X_\delta \to S_\bullet X$, and the crazy thing is that this actually geometrically realizes to a homotopy equivalence. In fact, we have $X = C_0 X \to S_\bullet X$ (given by constant simplices), and we claim that this is a weak equivalence – that is, that *in each degree* it's a homotopy equivalence. Dimitar will prove for us that (under certain assumptions) this guarantees that the realizations are equivalent, too.

Now, we have the diagram $X = S_0X \xrightarrow{\sim} S_\bullet X \xleftarrow{\text{id}} S_\bullet X_\delta$. What's crazy is that this is the only way to get between $S_0X$ and $S_\bullet X_\delta$; there's no map from a space to its discretization. However, let's apply the path space functor $Path(Y) = C^0(I, Y)$ (whose $k$-simplices are $Path(C^0(\Delta^k, X))$). Then we get $Path(S_\bullet X) \xleftarrow{\sim} Path(X) \overset{\sim}{\twoheadrightarrow} X$ (with the second map given by $ev_1$). But note that in the compactly generated topology, we have a homeomorphism $Path(C^0(\Delta^k, X)) \cong C^0(\Delta^k \times I, X)$. In this target, there is the subspace $C^0(\Delta^{k+1}, X) \subseteq C^0(\Delta^k \times I, X)$ given by taking those maps off the cylinder that map constantly off of $\Delta^k \times \{0\}$. So this inclusion is also an equivalence. On the other hand, if you follow through how the simplicial maps work, the bottom face $\Delta^k \times \{0\}$ collapses down to the 0-vertex of $\Delta^{k+1}$, and so this subspace – thought of as a simplicial space – is homeomorphic to the *simplicial* pathspace $P(S_\bullet X)$. Thus we have the diagram

$$\begin{tikzcd}
\Delta^{op} \arrow[r, "P(S_\bullet)"] \arrow[d, "\mathcal{P}^{op}"'] & \texttt{Top} \\
\Delta^{op}, \arrow[ur, "S_\bullet"'] &
\end{tikzcd}$$

where $\mathcal{P} : \Delta \to \Delta$ is given by $[n] \mapsto [n+1]$, and $(\sigma : [n] \to [m]) \mapsto (\mathcal{P}\sigma : [n+1] \to [m+1])$ is given by setting $\mathcal{P}\sigma(0) = 0$ and by shifting the rest over by one. There is a small lemma that $|S_0| \xrightarrow{\sim} |P(S_\bullet)|$; note too that we have a natural transformation $d_0^* : PS_\bullet \to S_\bullet$ (which we should think of as an evaluation map). This should help us understand the coming talk.

And now, on to Dimitar's talk!

## 7.1 Recollections

The theory of $(\infty, 0)$-category is as follows.

- An $(\infty, 0)$-category is a space that's homotopy equivalent to a CW-complex.
- An equivalence is a homotopy equivalence.
- The homotopy category of an $(\infty, 0)$-category is its fundamental groupoid.

Then, we continued by induction to define $(\infty, n)$-categories, as follows.

- An $(\infty, n)$-category $X$ is an $((\infty, n-1)\text{-}\texttt{Cat})$-enriched simplicial set, i.e.:
  - a set $X_0$ of objects, and
  - for each $(x_0, \ldots, x_k) \in (X_0)^{k+1}$, an $(\infty, n-1)$-category $X(x_0, \ldots, x_k)$, such that
  - the Segal maps
  $$\sigma_n : X(x_0, \ldots, x_n) \to X(x_0, x_1) \times \cdots \times X(x_{n-1}, x_n)$$
  are equivalences of $(\infty, n-1)$-categories.

  (Note that we're leaving out some things, like for instance that the morphism space associated to a single object is contractible; this is why the above string of products doesn't need to be decorated as fiber products.)

We then have the following special case. Every $(\infty, 0)$-category is an $(\infty, 0)$-groupoid. Then, an $(\infty, n)$-groupoid is an $(\infty, n)$-category such that for all $(x_0, \ldots, x_k) \in (X_0)^{k+1}$, $X(x_0, \ldots, x_k)$ is an $(\infty, n-1)$-groupoid. Given an $(\infty, n)$-groupoid $X$, we have its homotopy category $h^n(X)$, which is a groupoid in the ordinary sense.

## 7.2 The lemma

We saw the following theorem last time.

**Theorem 4.** *Given an $(\infty, 1)$-groupoid $X$, the map $X(a, b) \to P_{a,b}|X|$ is a weak equivalence for all $(a, b) \in (X_0)^2$.*

This implies Peter's claim, since we defined the homotopy groups of the $(\infty, n)$-groupoid $X$ in terms of the spaces $X(a, b)$, whereas the homotopy groups of the space $P_{a,b}|X|$ are just shifted homotopy groups of $|X|$.

This was originally Segal's result, but he did it much before $(\infty, n)$-categories were around. So instead, he proved this theorem in the special case that $X_0 = \{*\}$. We'll restrict ourselves to this special case.

Segal was actually interested in this for different reasons. He was contributing to the so-called *delooping machine*. The question is, what are the conditions on $X$ such that we have an equivalence $X \simeq \Omega Y$? This arises here because there's a unique morphism space when $X_0 = \{*\}$. In fact, there's a very beautiful characterization of loopspaces, that they are precisely the spaces equipped with the structure that makes them into such morphism spaces!

We will actually prove the following slight generalization.

**Theorem 5.** *Let $X : \Delta^{op} \to \texttt{Top}$ be a simplicial space, such that:*

1. *$X_0 = \{*\}$ and $X_1$ is connected;*
2. *The maps*
$$\sigma_m = ((f_1^*, f_2^*, \dots, f_n^*) : X_n \to X_1 \times \cdots \times X_1$$
*(where $f_u : [1] \to [n]$ is given by $f_i(0) = i - 1$ and $f_i(1) = i$ – this is the spine map) are homotopy equivalences.*

*Then the map $X_1 \to \Omega|X|$ (adjoint to $\Sigma X_1 \to |X|$) is a homotopy equivalence iff $X_1$ has a homotopy inverse.*

Note that $X_1$ is an H-space via $\sigma_2 : X_2 \xrightarrow{\simeq} X_1 \times X_1$; we can choose a homotopy inverse to this, and then the map $m : X_2 \to X_1$ (given by $m(0) = 0$ and $m(1) = 2$) gives us a multiplication. Of course, since we chose a homotopy inverse, there's no hope of this having any sort of nice associativity properties or anything like that.

We also explain the map $\Sigma X_1 \to |X|$. This is really just the inclusion of the 1-simplices; normally this would be a map $\Delta^1 \times X_1 \to |X|$, but since we've declared that $X_0$ is just a point, this factors through (any given explicit model for) the suspension.

*Proof.* The "only if" is obvious: the inverse on a loopspace is just given by running loops backwards. So we can focus on the "if" direction.

Let $P(X) : \Delta^{op} \to \texttt{Top}$ be the simplicial path space, i.e. $P(X) = X \circ \mathcal{P}$ where $\mathcal{P} : \Delta \to \Delta$ is as Peter described. This gives that $(P(X))_n = X_{n+1}$, and $d_0 : [n] \to [n+1]$ induces $d_0^* : PX_n = X_{n+1} \to X_n$, i.e. the natural transformation $ev_1$.

To prove the statement, we will construct a diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & {|PX|} \arrow[d] \\
* = X_0 \arrow[r] & {|X|}
\end{tikzcd}$$

and show that it is homotopy cartesian. This will imply our claim since it will imply that we can take any factorization the map $|PX| \to |X|$ as $|PX| \overset{\sim}{\hookrightarrow} |PX|' \twoheadrightarrow |X|$ and then take the actual pullback (i.e. the fiber) of $PX' \twoheadrightarrow |X|$, and the map from $X_1$ will be an equivalence. But of course, $\Omega|X| = \text{fiber}(P_{x_0}|X| \to |X|)$, and so if we choose $|PX|'$ to be $P_{x_0}|X|$, then we get $X_1 \xrightarrow{\sim} \Omega|X|$.

So, how do we show that this diagram is homotopy cartesian? We actually prove this using the following more general statement.

**Proposition 9.** *Let $X, X' \in \mathtt{Top}^{\Delta^{op}}$, and let $f : X' \to X$ be a simplicial map. If for all morphisms $\theta : [m] \to [n]$ the diagram*

$$\begin{tikzcd}
X'_n \arrow[r, "\theta^*"] \arrow[d, "f_n"'] & X'_m \arrow[d, "f_m"] \\
X_n \arrow[r] & X_m
\end{tikzcd}$$

*is homotopy cartesian, then the diagram*

$$\begin{tikzcd}
|\Delta^n| \times X'_n \arrow[r] \arrow[d] & |X'| \arrow[d] \\
|\Delta^n| \times X_n \arrow[r] & |X|
\end{tikzcd}$$

*is homotopy cartesian for all $n$.*

This will prove our theorem if we take $X' = PX$ (the simplicial pathspace), then we can prove that the diagrams induced by $\theta$ will indeed be homotopy cartesian, and taking $n = 0$ will recover the main diagram in the proof.

Let us now show why the diagrams induced by $\theta$ are homotopy cartesian. We claim that it suffices to check this statement for maps of the form $\theta : [0] \to [n]$ given by $\theta(0) = n$. To see what's going on, let's take $n = 1$. Then our diagram becomes

$$\begin{tikzcd}
PX_1 = X_2 \simeq X_1^2 \arrow[r, "PX(\theta)"] \arrow[d, "d_0^* = pr_2"'] & PX_0 = X_1 \arrow[d, "d_0^*"] \\
X_1 \arrow[r, "X(\theta)"'] & X_0 = *.
\end{tikzcd}$$

Recall that $PX(\theta) = (X \circ P)(\theta)$, and that $P : \Delta \to \Delta$ is given on objects by $[n] \mapsto [n+1]$ and on morphisms by setting $(P(\theta))(0) = 0$ and shifting everything else by 1. So in this case, $P(\theta) : [1] \to [2]$ is given by $0 \mapsto 0$ and $1 \mapsto 2$. Then, $X(P(\theta)) = m$ (the mapwe defined above). And now we must assume that we have a homotopy inverse. We claim that if $X$ is an H-space with multiplication $m : X \times X \to X$, then the "shearing" map $Sh : X \times X \to X \times X$ given by $(x, y) \mapsto (m(x, y), y)$ is a homotopy equivalence iff $m$ has a homotopy inverse $i : X \to X$ (which wlil then we given by $m(\text{id}, i) \simeq \text{id}$). So if $m$ (as in the theorem) has a homotopy inverse, we get a diagram

$$\begin{tikzcd}
X_2 \arrow[dr, "\simeq_{Sh}"] & & \\
& X_1 \times X_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
& X_1 \arrow[r] & X_0
\end{tikzcd}$$

factoring the previous one. If we take an arbitrary $\theta$, the statement follows from the so-called "pullback lemma". First, things are quite similar if we have $\theta : [0] \to [n]$, and then arbitrary $\theta : [m] \to [n]$ are tackled by considering the diagram

$$\begin{tikzcd}
X_1^{n+1} \arrow[r] \arrow[d] & X_1^{m+1} \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_1^n \arrow[r] & X_1^m \arrow[r] & X_0;
\end{tikzcd}$$

since the outer rectangle and the right square are homotopy pullbacks, then the left square is a homotopy pullback too.

So, it remains to prove the proposition.

*Proof of proposition.* We'll ignore the difference between the ordinary realization $|X|$ of a simplicial space $X$ and the "fat" realization $||X||$, where doesn't quotient out by degeneracies. (They are compared very nicely in the appendix to Segal's paper.) This is legal because $X_0 \simeq *$.

We go by induction to show that for all $m$, the diagram

$$\begin{tikzcd}
|\Delta^m| \times X'_m \arrow[r] \arrow[d] & ||X'||_m \arrow[d] \\
|\Delta^m| \times X_m \arrow[r] & ||X||_m
\end{tikzcd}$$

is homotopy cartesian.

Note that $||X||_m$ is homeomorphic to the double mapping cylinder of $||X||_{m-1} \leftarrow |\partial\Delta^m| \times X_m \to |\Delta^m| \times X_m$. We use without proof the following fact: if we have a diagram of spaces

$$\begin{tikzcd}
Y_1 \arrow[d] & Y_0 \arrow[l] \arrow[r] \arrow[d] & Y_2 \arrow[d] \\
X_1 & X_0 \arrow[l] \arrow[r] & X_2
\end{tikzcd}$$

such that the two small squares are homotopy cartesian, then the induced map on double mapping cylinders

$Y \to X$ fits into homotopy cartesian diagrams

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] & Y \arrow[d] \\
X_i \arrow[r] & X
\end{tikzcd}$$

for $i = 0, 1, 2$.

We're running out of time, so we just sketch the idea from here on out. Each $||X||_m$ is a double mapping cylinder as we have already described, so $||X||$ is equivalent to the mapping telescope of $||X||_0 \to ||X||_1 \to ||X||_2 \to \cdots$. Each of the diagrams producing the $||X||_m$ is homotopy cartesian, and ultimately this shows that the mapping telescope itself is homotopy cartesian. The inductive assumption comes into play in the above unproved fact: one of the squares is only homotopy cartesian by assumption.

□

This completes the proof of the theorem. □

# 8 The model structure on Segal $n$-categories – Piotr Prstagowski

Peter gives a very brief introduction: "Now we're moving into a new part of the seminar, where now we try to get a handle on *morphisms* of higher categories." And now, on to Piotr's talk!

## 8.1 Internal hom and relative categories

We begin with the internal hom. Recall that in `Sets`, we have $\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, Z^Y)$, where we simply define $Y^Z = \mathrm{Hom}(Y, Z)$. This motivates the following definition.

**Definition 29.** Let $C$ be a category and $X \in \mathrm{ob}(C)$. If there exists a right adjoint to the functor $A \mapsto A \times X$, we call it the *internal Hom* from $X$, and denote it by $A \mapsto [X, A]$.

Note that this may not exist for all objects $X$.

**Proposition 10.** *Let $C$ be such that all internal homs exist. Then they are all compatible, i.e. they glue into a bifunctor $[-,-] : C^{op} \times C \to C$.*

*Proof.* Yoenda lemma. □

**Example 19.** The categories `Sets`, `sSets`, and `Spaces` all have internal homs.... almost. Actually, this is precisely the reason that most people restrict to *compactly generated* spaces. For instance, in `sSets`, $([X, Y])_k = \mathrm{Hom}(X \times \Delta^k, Y)$. The categories `Ab`, $\texttt{Mod}_R$, and $\texttt{Ch}(R)$ also all admit internal homs (although the monoidal structures there must be the appropriate tensor products, not Cartesian products.)

**Definition 30.** A *relative category* is a pair $A' \subseteq A$ of categories such that the inclusion induces a bijection $\mathrm{ob}(A') \simeq \mathrm{ob}(A)$.

Since we really do mean to demand that $A'$ contains all the objects of $A$, then this is equivalent to simply choosing a subclass of morphisms which is closed under composition.

**Example 20.** There is the relative category $(\texttt{nCat}, \sim) \subseteq (\texttt{nCat})$.

The motivation for this is that we would often like to consider $A'$ as consisting entirely of "isomorphisms". That is, we would like to obtain a *space* of morphisms in $A$ which allows for zig-zags where the backwards maps live in $A'$. We will not actually define these mapping spaces – that's left for a different talk – but we'll focus on the resulting internal hom.

Here is the central theorem of this talk.

**Theorem 6.** `nCat` *admits an internal hom.*

(When we say `nCat`, we either mean weak $n$-categories or $(\infty, n)$-categories.)

*Proof sketch.* There is an embedding $\texttt{nCat} \subseteq \texttt{Fun}((\mathbf{\Delta}^{op})^{\times n}, \texttt{Sets}) = \texttt{Psh}(\mathbf{\Delta}^{\times n})$, as follows. Given $F, G \in \texttt{Psh}(\mathbf{\Delta}^{\times n})$, we define $[F, G]$ by $[F, G](\Delta^{k_1} \times \cdots \times \Delta^{k_n}) = \mathrm{Hom}(F \times (\Delta^{k_1} \times \cdots \times \Delta^{k_n}), G)$. $\square$

We're pretty sure this construction actually works in all functor categories – not just presheaf categories. Also, this is what we already saw for `sSets`.

Now, there exists a certain $n$-category $S^{k_1,\ldots,k_n}$ such that for $C, D \in \texttt{nCat}$, $[C, D]_{\texttt{nCat}}$ is given by $[C, D]_{\texttt{nCat}}(\Delta^{k_1} \times \cdots \times \Delta^{k_n}) = \mathrm{Hom}(C \times S^{k_1,\ldots,k_n}, D)$. The existence of this object is something of a piece of folklore, though.

One thing to observe here is that $D$ is an $n$-category, and to understand its objects we just need to set all $k_i = 0$. So note that this really contains more informations than the hom-sets. On the other hand, Kim showed us that there may not be "enough functors" between $n$-categories in general. Moreover, this construction does not respect equivalences of $n$-categories.

The solution to these problems is the following. We find a certain subcategory of `nCat` with "nice" objects. Here, "nice" will be defined by the following theorem.

**Theorem 7.** *There exists a full subcategory* $\texttt{nCat}^{fib} \subset \texttt{nCat}$ *such that:*

1. *Every $n$-category is equivalent to one in* $\texttt{nCat}^{fib}$.
2. *Every equivalence* $D \xrightarrow{\sim} D'$ *in* $\texttt{nCat}^{fib}$ *induces an equivalence* $[C, D] \xrightarrow{\sim} [C, D']$.

We call this the subcategory of *fibrant objects.*

**Example 21.** In the case $n = 0$, we can consider `nCat` = `sSets`, and then this is the subcategory of Kan complexes.

**Remark 10.** This terminology comes from the fact that there is a model structure on a certain larger category containing `nCat`. This happens often: our original category may not be co/complete, but we can embed it into one that is, and then look for a model structure there. In fact, the specific embedding we take is of `nCat` into $(n-1)$`Cat`-enriched simplicial sets (i.e., the category of *precategories* enriched in $(n-1)$`Cat`). The general statement is that if $M$ is a nice (left proper, tractable, cartesian) model category, then $PC(M)$ is also a nice model category. Here, a fibrant object $X \in PC(M)$ will necessarily satisfy the Segal condition and have all $X(x_0, \ldots, x_n)$ as well. For more, check out "Homotopy theory of higher categories" by Simpson (Chapters 9-22).

**Remark 11.** In many of these examples, everything is cofibrant. This is why we are only focusing on the fibrant ones.

**Remark 12.** Actually, $\texttt{nCat}^{fib} \subseteq \texttt{nCat}$ extends to a morphism of relative categories, and (in an appropriate sense) this will be an equivalence.

**Remark 13.** It might seem weird that we're looking at model structures – i.e., $(\infty, 1)$-category structures – on `nCat`. However, internal homs capture all the higher (noninvertible) morphisms that we might otherwise be ignoring, and so we really won't be losing anything after all.

## 8.2 Description of fibrant $n$-categories

We will need to describe:

- equivalences;
- fibrant objects;
- fibrations $C \twoheadrightarrow D$ where $D$ is fibrant.

We need the following notation. Let $E$ be the 1-category with two objects with a single isomorphism between them. Then we write $\overline{E}$ for $E$, considered as an $n$-category. Also, if $X \in \mathrm{ob}(\mathtt{nCat})$ has object set $S$, then we write

$$X_n = \coprod_{(s_0,\ldots,s_n)\in S^{n+1}} X(s_0,\ldots,s_n).$$

Now, we can begin quite easily. At $n = 0$ we take `Sets` to have the unique model structure where the isomorphisms are the weak equivalences; all morphisms will be both fibrations and cofibrations. (This is the only model structure which doesn't collapse the homotopy category to a single point.) Then at $n = 1$ we take `sSets` with the Quillen model structure. This generalizes as follows.

Let $n > 0$. Then we define weak equivalences to be equivalences between categories (as we have defined previously). Now, suppose $F : C \to D$ is a functor of $n$-categories and $D$ is fibrant. Then we say that $F$ is a *fibration* iff:

1. *[isofibration]* it has the right lifting property with respect to the morphism $\{*\} \hookrightarrow \overline{E}$ of $n$-categories;
2. *[Segal]* the map $C_n \to C^1 \times_{C^0} \cdots \times^{C^0} C^1 \times_{D^1\times_{D^0}\cdots\times_{D^0} D^1} D^n$ induced by the diagram

$$\begin{tikzcd}
C_n \arrow[r] \arrow[d] & D_n \arrow[d] \\
C^1 \times_{C^0} \cdots \times_{C^0} C^1 \arrow[r] & D^1 \times_{D^0} \cdots \times_{D^0} D^1
\end{tikzcd}$$

   is a trivial fibration (i.e. a fibration and a weak equivalence (of $(n-1)$-categories)).

3. *[Reedy]* the morphism $C^n \to C(\partial\Delta^n) \times_{D(\partial\Delta^n)} D^n$ induced by the diagram

$$\begin{tikzcd}
C^n \arrow[r] \arrow[d] & D^n \arrow[d] \\
C(\partial\Delta^n) \arrow[r] & D(\partial\Delta^n)
\end{tikzcd}$$

   is a fibration.

Note that this tells us what the fibrant objects are, just by checking unique morphisms to the terminal $n$-category. Of course, the Segal condition is referring to the "spine inclusion" that we've seen before. For the Reedy condition, note that the boundary of a simplex is defined as a coequalizer, so this is defined as an equalizer.

**Remark 14.** Let's look at the isofibration condition for ordinary categories. Now $F : C \to D$ is a functor of 1-categories, and we assume we have a diagram

$$\begin{tikzcd}
\{*\} \arrow[r] \arrow[d, hook] & C \arrow[d, "p"] \\
E \arrow[r] & D.
\end{tikzcd}$$

The upper functor chooses an object $c \in \mathrm{ob}(C)$, and the lower functor chooses an object $d \in \mathrm{ob}(D)$ and an isomorphism $h : p(c) \xrightarrow{cong} p(c)$. So, we have a fibration iff this lifts to an isomorphism in $C$. (This is sometimes called the "folk" model structure on categories.) The other two conditions will be empty in the case of $n = 1$.

## 8.3 Back to 2-categories

Let's return to Kim's example back in 2-categories. Let $G$ be a group, seen as a groupoid, seen as a 2-category (i.e. with only identity 2-morphisms). Let $A$ be an abelian group, seen as the 2-morphisms $\mathrm{Aut}(\bullet \to \bullet)$ (i.e. $A$ is a 2-category). Recall that there's only a single strict functor between these two 2-categories. However, we have that $BG \simeq K(G, 1)$ and $BA \simeq K(A, 2)$ (since the 1-category $(\bullet \to \bullet)$ is contractible). So as we've defined it, $[G, A] \not\simeq [K(G, 1), K(A, 2)] = H^2_{grp}(G; A)$. This failure comes from the fact that $A$ is not fibrant. What Kim was telling us is that if we see $A$ as a bicategory, then there's a nicer way to see this: its 2-nerve $NA$ is fibrant.

Let's actually check this. To show that $A$ isn't fibrant, there are two possible lines of attack. We could look at the Segal condition, but in fact these are isomorphisms so we won't get anywhere. Instead, we check the Reedy condition. Namely, we would like to show that $A_2 \to A(\partial\Delta^2)$ is not a fibration. So first of all, note that

$$A_2 = A(0, 0) \times A(0, 1) \sqcup A(0, 0) \times A(0, 0) \sqcup \cdots ,$$

whereas

$$A(\partial\Delta^2) = A(0, 0) \times A(0, 1) \times A(0, 1) \sqcup \cdots .$$

(This just comes from the obvious inclusion $A(\partial\Delta^2) \subseteq (A_1)^3$ defined by some gluing conditions.) We only look at the part $A(0, 0) \times A(0, 1) \subseteq A_2$; this gets mapped into $A(0, 0) \times A(0, 1) \times A(0, 1) \subseteq A(\partial\Delta^2)$. This sends a spine

$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[ur, "g\circ f"] \arrow[rr, "f"'] & & \bullet \arrow[ul, "g"']
\end{tikzcd}$$

to the evident component of $A(\partial\Delta^2)$. Note that the only composable arrows we have in $A(0, 0) \times A(0, 1)$ is just $(\mathrm{id}_0, f)$ (where $f$ is the unique nonidentity 1-morphism in $A$). This is not an isofibration: there are strictly more isomorphisms in the target than in the source.

So, when we "push the 2-cells of $A$ down" when we take a fibrant replacement, this gives us the correct $[G, NA]$ that we would've expected.

## 8.4 Further comments

Karol shows us inductively why these are fibrant. Given a simplicial set $K$, we can consider the evaluation $C(K)$. Then, for fixed $C$, this gives us a functor $\texttt{sSet}^{op} \to \texttt{nCat}$, the unique such one taking colimits to limits and satisfies $C(\Delta[n]) = C_n$. In particular, $C(\partial\Delta[n])$ is called a *matching object*; it comes with a canonical map $C_n = C(\Delta[n]) \to C(\partial\Delta[n])$. In fact, whenever we have an inclusion $K \hookrightarrow L$ of ssets, then $C(L) \to C(K)$ is a fibration. The way we do this is write out the inclusion as iterated pushouts along boundary inclusions (i.e., we attach all nondegenerate simplices of $L$ which are not in $K$). This flips via $C(-)$ to a sequence of fibrations.

Now, $D_n = D(\Delta[n]) \to D_1 \times_{D_0} \cdots \times_{D_0} D_1 = D(S[n])$, where $S[n]$ denotes the spine. Since $S[n] \hookrightarrow \Delta[n]$, we see that this map must be a fibration. From here, we can compose all the way down to $\mathrm{pt} = D(\emptyset)$. The inclusion of $\emptyset$ is of course always a cofibration, so this map must be a fibration. Now, in the Segal diagram if we take a pullback then by what we have seen, the pullback will also be fibrant.

We note that in Piotr's definition of a fibration with fibrant target, assuming everything works out in the end, we could've assumed that the source is also fibrant.

Now, Chris says a few words. First of all, the argument that Karol described relies on having a pullback of a fibration be a fibration. In fact, we also have this by our $n$-categorical fibration by induction.

We also give an example of this $n$-category $S^{k_1,\dots,k_n}$ through which we define the inner hom. Let's look at the case $n = 2$, which is the easiest case where we can see something interesting happening. So, corresponding to any $\Delta^{k_1} \times \Delta^{k_2}$ is a strict 2-category given by a grid: there are $k_1 + 1$ objects, with $k_2$ morphisms going across from each object to the next. This is our $S^{k_1,k_2}$. Now, $\mathtt{2Cat} \subseteq \mathtt{Fun}(\mathbf{\Delta}^{op}, \mathtt{1Cat}) \subseteq \mathtt{Fun}(\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}, \mathtt{Set})$, and this last category has an easy internal hom. Moreover, if $X$ and $Y$ are 2-categories, then this internal hom will again be a 2-category, and will in fact be the correct internal hom there. Namely, we claim that if $Y \in \mathtt{Fun}(\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}, \mathtt{Set})$ comes from a 2-category then there is a unique dotted arrow in the diagram

$$\begin{tikzcd}
\mathbf{\Delta}^{k_1} \times \mathbf{\Delta}^{k_2} \arrow[r] \arrow[d] & Y \\
S^{k_1,k_2}. \arrow[ur, dashed] &
\end{tikzcd}$$

At the suggestion/questioning of Karol, we look at the case where our category has discrete objects. We have the multi-simplicial object $Y$, where $Y_0$ is discrete (so $Y_{0i} = Y_{00}$ for all $i$), $Y_1 = (Y_{10} \leftleftarrows Y_{11} \Leftarrow Y_{12} \cdots)$, and similarly for $Y_2$. This is what we get for $\mathbf{\Delta}^0 \times \mathbf{\Delta}^k \to Y$. But by the adjunction, this should give us a map $\mathbf{\Delta}^k \to Y_0$. So what we're claiming is the existence of a unique filler in

$$\begin{tikzcd}
\mathbf{\Delta}^{0} \times \mathbf{\Delta}^{k} \arrow[r] \arrow[d] & Y \\
\mathbf{\Delta}^{0} \times \mathbf{\Delta}^{0}. \arrow[ur, dashed] &
\end{tikzcd}$$

Now, for a bigger grid associated to $\mathbf{\Delta}^{k_1} \times \mathbf{\Delta}^{k_2}$, suppose we have a map into the grid presentation of $Y$. The source admits maps in from $\mathbf{\Delta}^0 \times \mathbf{\Delta}^{k_2}$ – in fact, one for each object. Namely, this picks out the associated column. Then we're taking a pushout to $\mathbf{\Delta}^0 \times \mathbf{\Delta}^0$, and the filler is asking for an object. But $Y$, since it comes from an $n$-category, satisfies the Segal conditions. And this is what allows us to build the category $S^{k_1,k_2}$ by crushing out the columns of $\mathbf{\Delta}^{k_1} \times \mathbf{\Delta}^{k_2}$ and then adding in all the necessary compositions. This gives us the right set $\mathrm{Hom}(X \times S^{k_1,k_2}, Y)$.

# TALKS AFTER I WILL HAVE LEFT

Let me know if you're willing to tex these!

## 9 Segal categories vs. quasicategories – ...

## 10 Relative categories vs. quasicategories – ...

## 11 The homotopy hypothesis – ...

## 12 On the relation to topological field theories – Peter Teichner