From homotopical algebra to homotopical algebraic geometry

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1.1 Relative algebraic geometry

By definition, a scheme is obtained by gluing together affine schemes. As affine schemes are themselves in one-to-one correspondence with rings[1], a scheme is also obtained by gluing rings together. This observation has lead to the following idea of *relative algebraic geometry*.

Main idea of relative algebraic geometry: (Grothendieck, Hakim, Deligne ...) *One can extend the notion of schemes by formally replacing rings with any kind of "ring-like object"..*

Here, a "ring-like object" is by definition a commutative monoid in a symmetric monoidal category $(C, \otimes)$. In other words, it consists of an object A in C, and morphisms $\mu : A \otimes A \longrightarrow A$ (multiplication), $e : \mathbf{1}_C \longrightarrow A$ (unit), satisfying the usual identities. Here are some examples of context of

[1]The expression *ring* will always mean commutative ring with unit.

relative algebraic geometry.

1. $C = \mathbb{Z}/2 - Vect$, the category of $\mathbb{Z}/2$-graded vector spaces (say over $\mathbb{C}$), equiped with the graded tensor product. Ring-like objects are then $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras (commutative in the graded sense), and they correspond to affine super-schemes. By gluing them together one gets the general notion of super-schemes.

2. Let G be a linear algebraic group (say over $\mathbb{C}$), and $C = Rep(G)$ be the category of linear representations of G (maybe of infinite dimension), endowed with the tensor products of representations. Ring-like objects are G-equivariant $\mathbb{C}$-algebras, which correspond to affine G-schemes (i.e. affine schemes with an action of G). By gluing one gets the notion of G-schemes. This example is a particular case of algebraic geometry over Tannakian categories used by Deligne.

3. Let $(S, \mathcal{O}_S^{an})$ be a complex analytic space, and $C = QCoh(S)$, the category of quasi-coherent sheaves of $\mathcal{O}_S^{an}$-modules on S. Ring-like objects are sheaves of quasi-coherent $\mathcal{O}_S^{an}$-algebras, and one thinks of them as "analytic families of affine schemes parametrized by S". By gluing, one obtains a notion of "analytic families of schemes parametrized by S", which were called by Hakim *relative schemes over S*.

4. According to Smirnoff, Kapranov, Soulé ... there should exist a useful notion of "schemes over the field with one element $\mathbb{F}_1$". The object $\mathbb{F}_1$ is of course undefined, but the category of $\mathbb{F}_1$-vector spaces have been declared to be the category Set_* of pointed sets. This is a symmetric monoidal category for the smash product (i.e. the categorical product in Set_*). Commutative monoids in Set_* are thought as "affine schemes over $\mathbb{F}_1$", and one could try to glue them to get a (useful ?) notion of "schemes over $\mathbb{F}_1$".

5. A bit out of our context, one can try to glue associative (exceptionaly non commutative) rings together, in order to define a notion of "non commutative schemes". This approach has already been used by Kontsevich, Rosenberg and Orlov.

Remark 1.1.1 In all of these examples, we did not describe the "way of gluing", or equivalentely the "topology" which is used. It is in general a non trivial problem to find intersting topologies (e.g. in the example (4)).

1.2 Homotopical algebraic geometry

Main idea of homotopical algebraic geometry ("HAG" for short): *One can extend furthermore the notion of schemes by formally replacing rings with any kind of "homotopy-ring-like object"..*

Here, a "homotopy-ring-like object" is a commutative monoid in a symmetric monoidal category endow with a notion of equivalences. This notion of equivalences is the new input of HAG, and some times "monoids" in this context will have to be understood as "up-to-homotopy monoid" (e.g. as E_∞-rings).

Standard examples:

1. cdga:=commutative differential graded algebras (over a field of carateristic 0). These are commutative monoids in the category of complexes (non positive, non negative or unbounded). The equivalences in this context are the quasi-isomorphisms.

2. E_∞-algebras. They are the correct generalization of the notion of cdga when the base field is not of caracteristic 0. Again, the equivalences are the quasi-isomorphisms.

3. E_∞-ring spectra. They are the commutative monoids in the category of spectra for the smash product. The equivalences are the stable weak equivalence.

4. Symmetric monoidal categories. They are the commutative monoids in the category of categories. The equivalences are the equivalences of categories.

We will now present one of the motivations for our interest in HAG. It is concerned with "derived algebraic geometry". As far as we know these ideas were introduced by Deligne, Drinfeld, Kontsevich and Kapranov.

Let C be a smooth projective curve (say over $\mathbb{C}$), and let us consider the moduli stack $\underline{Vect}_n(C)$ of rank n vector bundles on C (here $\underline{Vect}_n(C)$ classifies all vector bundles on C, not only the semi-stable or stable ones). The stack $\underline{Vect}_n(C)$ is known to be an algebraic stack (in the sense of Artin). Furthermore, if $E \in \underline{Vect}_n(C)(\mathbb{C})$ is a vector bundle on C, one can easily compute the *stacky tangent space* of $\underline{Vect}_n(C)$ at the point E. This *stacky tangent space* is actually a complex of $\mathbb{C}$-vector spaces concentrated in degrees $[-1, 0]$, which is easily seen to be quasi-isomorphic to the complex $C^*(C_{Zar}, \underline{End}(E))[1]$ of Zariski cohomology of C with coefficient in the vector bundle $\underline{End}(E) = E \otimes E^*$. Symbolically, one writes

$$T_E \underline{Vect}(C) \simeq H^1(C, \underline{End}(E)) - H^0(C, \underline{End}(E)).$$

This implies in particular that the *dimension* of $T_E\underline{Vect}(C)$ is independent of the point E, and is equal to $n^2(g-1)$, where g is the genus of C. The conlcusion is then that the stack $\underline{Vect}_n(C)$ is smooth of dimension $n^2(g-1)$.

Let now S be a smooth projective surface, and $\underline{Vect}_n(S)$ the moduli stack of vector bundles on S. Once again, $\underline{Vect}_n(S)$ is an algebraic stack, and the stacky tangent space at a point $E \in \underline{Vect}_n(S)(\mathbb{C})$ is easily seen to be given by the same formula

$$T_E \underline{Vect}_n(S) \simeq H^1(S, \underline{End}(E)) - H^0(S, \underline{End}(E)).$$

Now, as $H^2(S, \underline{End}(E))$ might jump when specializing E, the dimension of $T_E\underline{Vect}(S)$, which $h^1(S, \underline{End}(E)) - h^0(S, \underline{End}(E))$, is not locally constant and therefore the stack $\underline{Vect}_n(S)$ is not smooth anymore.

The main idea of *derived algebraic geometry* is that $\underline{Vect}_n(S)$ is only the truncation of a richer object $\mathbb{R}\underline{Vect}_n(S)$, called the *derived moduli stack of vector bundles on S*. This derived moduli stack, whatever it may be, should be such that its *tangent space* at a point E is the *whole* complex $C^*(S, \underline{End}(E))[1]$, or in other words,

$$T_E \mathbb{R}\underline{Vect}_n(S) \simeq -H^2(S, \underline{End}(E)) + H^1(S, \underline{End}(E)) - H^0(S, \underline{End}(E)).$$

The dimension of its tangent space at E is then expected to be $-\chi(S, \underline{End}(E))$, and therefore locally constant. Hence, the object $\mathbb{R}\underline{Vect}_n(S)$ is expected to be *smooth*.

Remark 1.2.1 Another, very similar but probably more striking example is given by the moduli stack of stable maps. A consequence of the expected existence of the *derived moduli stack of stable maps* is the presence of a *virtual structure sheaf* giving rise to a *virtual fundamental class*. The importance of such constructions in the context of Gromov-Witten theory shows that the extra information contained in *derived moduli spaces* is very interesting and definitely geometrically meaningful.

In the above example of the stack of vector bundles, the tangent space of $\mathbb{R}Vect_n(S)$ is expected to be a complex concentrated in degree $[-1,1]$. More generally, one can get convinced that tangent spaces of derived moduli (1-)stacks should be complexes concentrated in degree $[-1,\infty[$.

The important conclusion is that "tangent spaces of derived moduli spaces" should be complexes. Now, a smooth variety X locally at a point $x \in X$ looks like $Spec(Sym(T^*_{X,x}))$, where $T^*_{X,x}$ is the dual to the tangent space of X at x. Following the same principle, locally derived moduli spaces should look like $Spec(Sym(C))$, but now C is a complex of vector spaces. As $Sym(C)$ is a cdga we obtain the following expectation.

Expectation: *Local rings of derived moduli spaces are expected to be cdga's.*

This gives an hint that derived algebraic geometry might be understood as algebraic geometry over cdga's, and is therefore part of HAG.

1.3 Overview of the construction

In order to motivate our construction we present here a construction of the category of schemes.

Any scheme X (we will assume all schemes to be quasi-separated) defines a functor of points

$$\begin{array}{cccc} X(-): & (Rings) & \longrightarrow & (Sets) \\ & A & \mapsto & Hom_{Sch}(Spec\, A, X). \end{array}$$

This defines a functor $X \mapsto X(-)$, from the category of schemes to the category $\underline{Hom}(Rings, Sets)$, of functors from the category of rings to the category of sets. It is an easy exercice to check that this functor is fully faithful, and induces a full embedding

$$Sch \hookrightarrow \underline{Hom}(Rings, Sets).$$

The image of this full embedding can be reconstructed as follows.

For any ring $A \in Rings$, one has the representable functor

$$\begin{array}{cccc} h_A: & (Rings) & \longrightarrow & (Sets) \\ & B & \mapsto & Hom_{Rings}(A, B). \end{array}$$

The Yoneda lemma implies that $A \mapsto h_A$ induces a full embedding

$$(Rings)^{op} \hookrightarrow \underline{Hom}(Rings, Sets).$$

Objects in the image of the functor $A \mapsto h_A$ will simply be called *affine schemes* (note that $Spec\, A(-) \simeq h_A$).

We make the following definitions.

1. A morphism $f: F \longrightarrow G$ in $\underline{Hom}(Rings, Sets)$ is representable if for any $A \in Rings$, and any $h_A \longrightarrow G$, the functor $F \times_G h_A$ is an affine scheme.
2. A representable morphism $f: F \longrightarrow G$ in $\underline{Hom}(Rings, Sets)$ is furthermore an open immersion if for any $h_A \longrightarrow G$ as above, the projection
$$F \times_G h_A \simeq h_B \longrightarrow h_A$$
induced a morphism $A \longrightarrow B$ which is a (Zariski) localization (i.e. $Spec\, B \to Spec\, A$ is an open immersion).

3. A functor $F : Rings \longrightarrow Sets$ is a sheaf (for the Zariski topology) if for each ring A the restriction of F on $(Spec\, A)_{Zar}$ is a sheaf.

Proposition 1.3.1 *A functor $F : Rings \longrightarrow Sets$ is a scheme (i.e. is isomorphic to some $X(-)$ for a scheme X) if and only if it satisfies the following conditions.*

1. *F is a sheaf*
2. *The diagonal morphism $F \longrightarrow F \times F$ is representable.*
3. *There exists rings A_i, and a morphism*

$$\coprod_i Spec\, A_i \longrightarrow F$$

which is an epimorphism of sheaves, and such that any morphism $Spec\, A_i \longrightarrow F$ is an open immersion.

Conclusion: *The category of schemes only depends on the category of rings together with its Zariski topology.*

This conclusion is the base of relative algebraic geometry, as formally one can then define schemes for any category of ring-like objects endowed with a topology. It will also be our base for developping HAG. However, the new feature of dealing with a non trivial notion of equivalences will imply some complications. We will need in particular to define correctly the Yoneda embedding in this new context, as well as a new notion of topology.

2 Lecture 3. Categories with equivalences and model categories

The purpose of this lecture is only to give a very brief introduction to the the ory of model categories. There are many good books and papers on model categories, some of them listed in the bibliography below, to which we will refer for proofs and details. We will *always* neglect almost all set-theoretic problems concerning universes, ...

2.1 Categories with equivalences and their localizations

There are many examples in which, given a category $\mathcal{C}$, there is a distinguished set of morphisms W in $\mathcal{C}$ that one would like to consider as invertible. Examples: $\mathcal{C}$= "category of complexes of modules over some commutative ring k", W = "quasi-isomorphisms" ; $\mathcal{C}$ = "category of topological spaces", W = "weak homotopy equivalences" or W_{H} = " maps inducing isomorphisms on some fixed cohomology theory H".
It is very well known that there always exists a formal way to invert the morphisms W in $\mathcal{C}$; more precisely, there exists a category $W^{-1}\mathcal{C}$ together with a functor $\ell : \mathcal{C} \longrightarrow W^{-1}\mathcal{C}$ sending W to isomorphisms and having the following universal property "any functor from $\mathcal{C}$ that sends W to isomorphisms factors through ℓ". So, not only maps in W becomes isomorphisms in $W^{-1}\mathcal{C}$ but this construction is initial among all functors with the same property. Such a category is called the *localization of $\mathcal{C}$ with respect to W*.

Example 2.1.1 Let $\mathcal{C} = \mathrm{Ab}$ (the category of abelian groups) and W be the maps with torsion kernel and cokernel. Then $Hom_{W^{-1}\mathcal{C}}(A, B) = Hom_{\mathrm{Ab}}(A \otimes \mathbb{Q}, B \otimes \mathbb{Q})$.

How does one describe the category $W^{-1}\mathcal{C}$?
First of all, one can assume that W contains all the isomorphisms. Then, consider the category with the same objects as $\mathcal{C}$ and with morphisms from x to y given by equivalence classes of (reduced) strings

$$x \longrightarrow \cdots \qquad \cdots \longrightarrow y$$

where consecutive arrows have opposite directions and backwards arrows are in W. Two reduced strings are equivalent if, after forcing them to the same *length* (= number of arrows between x and y) and to the same *pattern* of arrows, then they can be connected by a commutative diagram like

$$\begin{array}{ccccccccccccc}
x & \text{---} & z_{11} & \text{---} & z_{12} & \text{---} & \cdots & & \cdots & \text{---} & z_{1m} & \text{---} & y \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
x & \text{---} & z_{21} & \text{---} & z_{22} & \text{---} & \cdots & & \cdots & \text{---} & z_{2m} & \text{---} & y \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
x & \text{---} & z_{n-1,1} & \text{---} & z_{n-1,2} & \text{---} & \cdots & & \cdots & \text{---} & z_{n-1,m} & \text{---} & y \\
\| & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \| \\
x & \text{---} & z_{n,1} & \text{---} & z_{n,2} & \text{---} & \cdots & & \cdots & \text{---} & z_{n,m} & \text{---} & y
\end{array}$$

where the vertical arrows are all in W (and commutativity makes sense because all the strings involved have the same pattern). One can check that definining $W^{-1}\mathcal{C}$ to be this category (and $\ell : \mathcal{C} \longrightarrow W^{-1}\mathcal{C}$ the obvious functor), we have universally inverted morphisms in W, i.e. we have built the localization of $\mathcal{C}$ with respect to W (which is uniquely defined up to a unique equivalence of categories).

Exercise 2.1.2 Can you figure out a way to organize all the different commutative diagrams as the one above (of various sizes) into a simplicial set? If you can, you have almost defined the so called *hammock* localization of Dwyer-Kan (in any case, see [D-K2]).

2.2 Model categories. Examples

As one may easily guess, in absence of additional hypothesis on the pair $(\mathcal{C}, W)$, working with $W^{-1}\mathcal{C}$ is in general almost hopeless. Therefore, one has singled out various conditions on the pair $(\mathcal{C}, W)$ that make the corresponding localization more tractable. Among these conditions, there are the well known *calculus of left, right or bilateral fractions* conditions, for which we refer to [Ga-Zi] or [Sch]. It will suffice to recall that if $\mathcal{A}$ is an abelian category, $\mathrm{K}(\mathcal{A})$ denotes the category of complexes in $\mathcal{A}$ with morphisms given by homotopy classes of maps and $W \subset \mathrm{K}(\mathcal{A})$ consists of quasi-isomorphisms, then the pair $(W, \mathrm{K}(\mathcal{A}))$ admits a bilateral calculus of fractions; the corresponding localization $W^{-1}\mathrm{K}(\mathcal{A})$ is the derived category $\mathcal{D}(\mathcal{A})$ of the abelian category $\mathcal{A}$ and it's known to be (somewhat) tractable.
The axioms of a *model structure* on a category supply another way to get a tractable localization (though this is not their only aim). In fact, one consequences of them is to allow a kind of hybrid left/right calculus of fractions. But before explaining this, let us give the axioms (due to D. Quillen).

Definition 2.2.1 *A model category is a complete and cocomplete category M together with the following data*

- *three distinguished classes of maps in M,* $(W, \mathrm{Fib}, \mathrm{Cof})$ *(*weak equivalences*,* fibrations *and* cofibrations*);*

- *there are two functorial factorizations* $f \longmapsto (p, i')$, $f \longmapsto (p', i)$ *(i.e.,* $f = p \circ i'$ *and* $f = p' \circ i$*, functorially in* f*),*

subject to the following axioms

1. (2 out of 3) *If* (f, g) *are composeable arrows (i.e.* $f \circ g$ *exists), then all* f*,* g *and* $f \circ g$ *are in* W *if any two of them are;*

2. (Lifting) *Calling maps in* $W \cap \mathrm{Fib}$ *(resp., in* $W \cap \mathrm{Cof}$*) trivial fibrations (resp. trivial cofibrations), in any commutative solid arrows square*

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ y & \longrightarrow & y \end{array}$$

   in which p *is a fibration and* i *is a cofibration, a dotted arrow exists if either* p *is a trivial fibration or* i *is a trivial cofibration;*

3. (Retracts) *Any retract of a weak equivalence (resp. a fibration resp. a cofibration) is a weak equivalence (resp. a fibration resp. a cofibration);*

4. (Factorizations) *For the two functorial factorizations* $f \longmapsto (p, i')$*,* $f \longmapsto (p', i)$ *(i.e.,* $f = p \circ i'$ *and* $f = p' \circ i$*, functorially in* f*),* p *is a fibration,* i' *a trivial cofibration,* p' *a trivial fibration and* i *a cofibration.*

Remark 2.2.2 If one thinks of $M = \mathrm{Top}$ then the lifting axiom is just a way to build in the definition both the covering homotopy theorem and the homotopy extension theorem.

For a model category M, the *associated homotopy category* is the category $\mathrm{Ho}(M) := W^{-1}M$. Since M is complete and cocomplete, there is an initial object $\emptyset$ and a final object $*$; an object x in M is then called *fibrant* (resp. *cofibrant*) if the map $x \to *$ (resp. the map $\emptyset \to x$) is a fibration (resp. a cofibration). The full subcategory of fibrant (resp. cofibrant, resp. cofibrant and fibrant) objects in M, will be denoted by M_f (resp. M_c, resp. M_{cf}). The existence of functorial factorizations (applied to the maps $x \to *$ and $\emptyset \to x$), then gives us functors R and $Q : M \to M$, together with natural transformations $\mathrm{Id} \longrightarrow R$ and $Q \longrightarrow \mathrm{Id}$ which are objectwise weak equivalences. R is called the *fibrant replacement* functor and Q the *cofibrant replacement* functor, in M.
In general, given a complete and cocomplete category M endowed with a triple $(W, \mathrm{Fib}, \mathrm{Cof})$ it is definitely not an easy task to check whether this gives a model structure on M or not; especially hard is to check that there is an associated pair of functorial factorizations satisfying (4).

Remark 2.2.3

1. The axioms are *redundant* in the sense that if M is a model category then Fib (resp. Cof) is determined by the pair (W, Cof) (by the pair (W, Fib)) via the lifting axiom.

2. The axioms are *self-dual* in that if M is a model category then the opposite category M^{op} is a model category with the same class of weak equivalences and with cofibrations and fibrations interchanged.

Example 2.2.4

1. Let R be a (commutative) ring and $\mathrm{Ch}_+(R)$ the category of chain complexes of R-modules in non-negative degrees (or of cochain complexes in non-positive degrees). Then

- $W :=$ quasi-isomorphisms;
- Fib $:=$ degreewise surjections in degrees ≥ 1;
- Cof :=degreewise injections with degreewise projective cokernel,

are part of a model structure on $\mathrm{Ch}_+(R)$ (see [D-S]). The associated homotopy category $\mathrm{Ho}(M)$ is the *derived category* $D^+(R)$.

2. R as above and $\mathrm{Ch}(R)$ the category of not necessarily bounded complexes of R-modules. Then,

   - $W :=$ quasi-isomorphisms;
   - Fib $:=$ degreewise surjections;
   - Cof $:=$ maps with the left lifting property with respect to any surjective quasi-isomorphism,

   are part of a model structure on $\mathrm{Ch}(R)$, called the *projective model structure* on $\mathrm{Ch}(R)$ (see [Ho, §2.3]). In this model structure:

   - if a complex $X_\bullet$ is cofibrant, then each X_n is a projective R-module (the converse is true if $X_\bullet$ is bounded below). Therefore, if A is an R-module, any usual projective resolution $P_\bullet \longrightarrow A$ (in the sense of homological algebra) is a cofibrant replacement; therefore cofibrant replacements generalizes the idea of projective resolutions (hence the name of the model structure);
   - any complex is fibrant;
   - cofibrations are degreewise split inclusions, whose cokernel is cofibrant.

   There is also a "dual" model structure where

   - $W :=$ quasi-isomorphisms;
   - Cof $:=$ degreewise injections;
   - Cof $:=$ maps with the right lifting property with respect to any injective quasi-isomorphism,

   which is called the *injective model structure* on $\mathrm{Ch}(R)$ (see [Ho, §2.3]). In this model structure:

   - if a complex $X_\bullet$ is fibrant, then each X_n is an injective R-module (the converse is true if $X_\bullet$ is bounded above). Therefore, if A is an R-module, any usual injective resolution $A \longrightarrow I_\bullet$ (in the sense of homological algebra) is a fibrant replacement; therefore fibrant replacements generalizes the idea of injective resolutions (hence the name of the model structure);
   - any complex is cofibrant;
   - fibrations are degreewise split surjections, whose cokernel is cofibrant.

   In both cases, the associated homotopy category $\mathrm{Ho}(M)$ is the *(unbounded) derived category* of R.

3. Let k be a field of characetristic zero and $M := (\mathrm{cdga}_k^{\leq 0})$ the category of nonpositively graded, differential graded-commutative algebras over k (with differential of degree $+1$). Then

   - $W :=$ quasi-isomorphisms;
   - Fib $:=$ degreewise surjections in degrees < 0;
   - Cof $:=$ maps with the left lifting property with respect to any surjective quasi-isomorphism,

   are part of a model structure on M. In this model structure:

- if a cdga $A^\bullet$ is *quasi-free* (i.e. there exists a free non-positively graded k-module $V^\bullet$ and an isomorphism of graded k-algebras $A^\bullet \simeq F(V^\bullet)$, where $F(-)$ denotes the free graded commutative k-algebra functor) then it is cofibrant. The basic example is given by *Koszul complexes.* Let $\underline{f} := (f_1, ..., f_n)$ be elements in some commutative noetherian k-algebra R. The associated Koszul complex gives a map $K(\underline{f}) \longrightarrow R/(\underline{f})$ (where $K(\underline{f})$ is quasi-free) which is a cofibrant replacement iff $\underline{f}$ is a regular sequence;
- as a partial converse to the above, any cofibrant $A^\bullet$ admits a trivial fibration $p : F^\bullet \longrightarrow A^\bullet$ from a quasi-free cdga $F^\bullet$ (in particular, then, p has a section).

4. Let $M :=$ Top be the category of topological spaces. Then,

   - $W :=$ weak homotopy equivalences (i.e. maps inducing isomorphisms on all the π_i's, for any choice of base point);
   - Cof := injections;
   - Fib := maps with the right lifting property with respect to any injective weak homotopy equivalence,

   are part of a model structure on Top. Fibrations here are exactly *Serre fibrations* i.e. maps $p : X \longrightarrow Y$ for which the classical covering homotopy theorem holds: for any $n \geq 0$, a dotted arrow in the following commutative diagram exists

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ D^n \times I & \longrightarrow & Y \end{array}$$

5. Let $M :=$ SSet be the category of simplicial sets (i.e. the category of functors $\Delta^{op} \longrightarrow Set$, where Δ denotes the standard simplicial category). There is a pair of adjoint functors

$$SSet \underset{Sing}{\overset{|-|}{\rightleftarrows}} Top$$

   where the left adjoint $|-|$ is the geometric realization functor (that takes the representable $\Delta[n]$ to the standard n-simplex Δ^n in $\mathbb{R}^n$ and is then extended by requiring preservation of colimits) and the right adjoint $Sing$ is the singular complex functor. Then,

   - $W :=$ weak homotopy equivalences (i.e. maps f such that $|f|$ is a wek homotopy quivalence in Top);
   - Cof := monomorphisms;
   - Fib := maps with the right lifting property with respect to any monic weak homotopy equivalence,

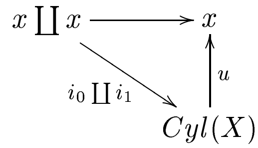
   are part of a model structure on SSet. Fibrations here are exactly *Kan fibrations* i.e. maps $p : X \longrightarrow Y$ which "lift on horns". More precisely, for $n \geq 0$ and any $0 \leq r \leq n$, let the r-th horn $\Lambda^r[n]$ be the sub-simplicial set of $\Delta[n]$ whose geometric realization is obtained from Δ^n by omitting the interior of Δ^n and the interior of the $(n-1)$-dimensional face opposed to the vertex r; then a Kan fibration is a map p such that for any $n \geq 0$ and any $0 \leq r \leq n$, a dotted arrow in the following commutative diagram exists

$$\begin{array}{ccc} \Lambda^r[n] & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta[n] & \longrightarrow & Y \end{array}$$

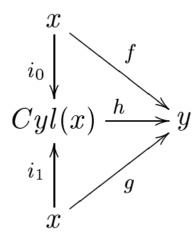
By a result of Quillen, the geometric realization functor preserves fibrations (and, by definition, weak equivalences).

2.3 The homotopy category of a model category

Let us see how the axioms of a model structure on M give an "easy" description of the homotopy category $\mathrm{Ho}(M) = W^{-1}M$.
If $x \in M$, a *cylinder* for x is a factorization



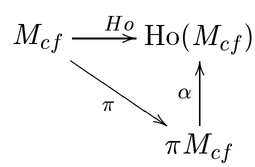
of the canonical map $x \coprod x \longrightarrow x$ into a cofibration $i_0 \coprod i_1$ followed by a weak equivalence u. The typical example is obviously $M = Top$ and $Cyl(X) = X \times I$. A cylinder for x in the opposite model category M^{op} is called a *cocylinder* or *path-object*, and denoted by $Cocyl(x)$. The factorization axiom ensures the existence of at least one (canonical) cylinder and cocylinder object for any $x \in M$.
As expected, (co)cylinders are designed to speak about *homotopies* between maps in M. If $f, g : x \longrightarrow y$ are maps in M a *left homotopy* is a map $h : Cyl(x) \longrightarrow y$ such that the following diagram commutes



If there exists a left homotopy between f and g, we write $f \sim^l g$. Dually, a *right homotopy* between f anf g is a left homotopy in the opposite model category; in this case, we write $f \sim^r g$. We write $f \sim g$ and say that f and g are *homotopic* if they are both left and right homotopic. A map $f : x \longrightarrow y$ is a *homotopy equivalence* if there exists a map $f' : y \longrightarrow x$, such that $ff' \sim id_y$ and $f'f \sim id_x$.

We list here the basic properties of left/right homotopies:

- If x is cofibrant, then $\sim^l$ is an equivalence relation in $Hom_M(x, y)$, for any $y \in M$;
- If y is fibrant, then $\sim^r$ is an equivalence relation in $Hom_M(x, y)$, for any $x \in M$;
- If x is cofibrant, y is fibrant and $f, g : x \longrightarrow y$, then $f \sim^l g$ iff $f \sim^r g$ iff $f \sim g$; therefore, $\sim$ is an equivalence relation in $Hom_M(x, y)$;
- Composition in M_{cf} is compatible with $\sim$. Hence there exists a quotient category $\pi M_{cf} := M_{cf}/\sim$ (whose isomorphisms are homotopy equivalences);
- If we consider the loclaization functor $Ho : M_{cf} \longrightarrow \mathrm{Ho}(M_{cf}) := W_{cf}^{-1}M_{cf}$, then $f \sim g$ implies $Ho(f) = Ho(g)$; therefore, there exists a factorization



- a map in M_{cf} is weak equivalence iff it is a homotopy equivalence; therefore there exists an induced functor $\beta : \mathrm{Ho}(M_{cf}) \longrightarrow \pi M_{cf}$;

- the functor $\alpha : \pi M_{cf} \longrightarrow \mathrm{Ho}(M_{cf})$ and $\beta : \mathrm{Ho}(M_{cf}) \longrightarrow \pi M_{cf}$ are mutually quasi-inverse;

The basic theorem that describes the homotopy category $\mathrm{Ho}(M)$ is the following

Theorem 2.3.1 *If M is a model category, then the natural inclusion $M_{cf} \longrightarrow M$ induces an equivalence of categories $\mathrm{Ho}(M_{cf}) \simeq \mathrm{Ho}(M)$ (whose quasi-inverse is induced by RQ). Therefore*

$$\mathrm{Ho}(M) \simeq \pi M_{cf}.$$

This gives a nice description of maps in the homotopy category $\mathrm{Ho}(M)$ as homotopy classes of maps in M:

$$[x,y] \equiv Hom_{\mathrm{Ho}(M)}(x,y) \simeq [RQx, RQy] \simeq Hom_M(RQx, RQy)/\sim .$$

Remark 2.3.2

- The functors $R : M \longrightarrow M_f$ and $Q : M \longrightarrow M_c$ induce equivalence of categories $\mathrm{Ho}(M_f) \simeq \mathrm{Ho}(M) \simeq \mathrm{Ho}(M_c)$.

- We can represent any morphism in $\mathrm{Ho}(M)$ from x to y as an equivalence class of strings in M of length 3

$$x \xleftarrow[u]{} x' \longrightarrow y' \xleftarrow[v]{} y$$

  where u and v are in W.

Exercise 2.3.3 Define $\pi^l M_c := M_c/\sim^l$ and $\pi^l W_c \subset \pi^l M_c$ tyhe image of $W \cap M_c$; analogous definition for $\pi^r M_f$ and $\pi^l W_f$. Prove that the pair $(\pi^l M_c, \pi^l W_c)$ (resp. $(\pi^r M_f, \pi^r W_f)$) has a calculus of left fractions (resp. of right fractions) and that the corresponding localization is $\mathrm{Ho}(M_c)$ (resp., $\mathrm{Ho}(M_f)$). So, $\mathrm{Ho}(M) \simeq \mathrm{Ho}(M_c) \simeq \mathrm{Ho}(M_f)$ can be computed using left or right fractions from $(\pi^l M_c, \pi^l W_c)$ or $(\pi^r M_f, \pi^r W_f)$.

2.4 Higher homotopical structures: homotopy limits, homotopy colimits and mapping spaces

So far we have seen that a model structure on a category induces a good description of its localization with respect to weak equivalences. However, the axioms imply a lot more *higher homotopical structure*. Before giving the full structure, let us have a quick look at the first stage. Let M be a model category, $x \in M_c$, $y \in M_f$ and $f, g : x \longrightarrow y$. If $h, h' : f \sim g$ are left homotopies, there is a similar notion of left homotopy between h and h': if $h : Cyl(x) \longrightarrow y$ and $h' : Cyl(x)' \longrightarrow y$, define a double cylinder as a factorization

$$\begin{array}{ccc} Cyl(x) \coprod_{x \coprod x} Cyl(x)' & \xrightarrow{h \coprod h'} & y \\ & \searrow^{i_0 \coprod i_1} & \uparrow u \\ & & Cyl_2(x) \end{array}$$

where u is a weak equivalence and proceed analogously. Similar constructions hold when replacing left homotopies with right homotopies. These constructions give equivalence relations in the set of left/right homotopies between f and g: the corresponding quotients are denoted by $\pi_1^l(x, y; f, g)$ and $\pi_1^r(x, y; f, g)$ and one can check that (since x is cofibrant and y is fibrant $\pi_1^l(x, y; f, g) \simeq \pi_1^r(x, y; f, g)$; therefore, we will denote any of this two sets simply as $\pi_1(x, y; f, g)$. For $f_1, f_2, f_3 \in Hom_M(x, y)$, and homotopies $h : f_1 \sim f_2$, $h' : f_2 \sim f_3$, have a natural composition (mimick the known construction for topological spaces!) $h * h'$ which is a homotopy between f_1 and f_3. This composition of homotopies is compatible with the equivalence realtion above, therefore we get an induced composition

$$\pi_1(x, y; f_1, f_2) \times \pi_1(x, y; f_2, f_3) \longrightarrow \pi_1(x, y; f_1, f_3).$$

Consider the category $\underline{\Pi}_1(x, y)$ whose objects are $Hom_M(x, y)$ and whose morphisms are given by $Hom_{\underline{\Pi}_1(x,y)}(f, g) := \pi_1(x, y; f, g)$, with the above composition. The upshot is that $\underline{\Pi}_1(x, y)$ is a *groupoid*, appropriately *functorial* in the pair (x, y) (see [Qui2], for details).

This is only the first step (actaully the 1-truncation) of a higher structure built "on" the Hom-sets of a model category: one can in fact define, for any $x, y \in M$, a simplicial set (whose homotopy type is well deined, i.e. the simplicial set is well defined in $\mathrm{Ho}(SSet)$ up to isomorphism) $Map_M(x, y)$, called the *mapping space* between x and y, such that

- $\pi_0(Map_M(x, y)) \simeq [x, y] \equiv Hom_{\mathrm{Ho}(M)}(x, y)$ and
- $\pi_1(Map_M(x, y); f) \simeq Aut_{\underline{\Pi}_1(x,y)}(f)$, if x is cofibrant and y is fibrant.

Let us follow a path to mapping spaces which is slightly different from the usual one (it might be called a "Derivateurs"-approach, see [He, Gr2, Mal]).
As we introduced them, model structures are tools to study *pairs* $(\mathcal{C}, W)$ of a category plus a distinguished set of arrows in it. Actually, the structure we are interested in is the triple $(\mathcal{C}, W; W^{-1}\mathcal{C})$, meaning that we study pairs as above with the aim of considering the corrseponding localizations. In studying such pairs (or triples), we want also to perform constructions on them, in particular considering *diagrams* and *limits/colimits*. If $I \in Cat$ is an "index" category, we may associate to any pair $(\mathcal{C}, W)$ the pair $(\mathcal{C}^I, W_I)$, where $\mathcal{C}^I$ is the category of functors $I \longrightarrow \mathcal{C}$ and W_I consists of those natural transformations which are objectwise in W. The constant functor $c : \mathcal{C} \longrightarrow \mathcal{C}^I$ gives an map of pairs $(\mathcal{C}, W) \longrightarrow (\mathcal{C}^I, W_I)$ and therefore induces a functor

$$c : W^{-1}\mathcal{C} \longrightarrow W_I^{-1}(\mathcal{C}^I).$$

As usual in category theory, at this point one asks whether this functor c has left/right adjoints. A left (resp. right) adjoint (when it exists) is called the *homotopy colimit* along I and denoted by $hocolim_I$ (resp. the *homotopy limit* along I and denoted by $holim_I$.

Remark 2.4.1 Why have we considered $W_I^{-1}(\mathcal{C}^I)$ instead of $(W^{-1}\mathcal{C})^I)$, i.e. why have we looked for new concepts of limits/colimits? There is a conceptual answer to this: from the point of view of studying the category of pairs $(\mathcal{C}, W)$ as above, "diagrams" should give rise again to pairs and with this diagrams at hand we pass to the corresponding localizations and ask the familiar questions that usually lead to the concepts of limits/colimits. This is what we have done above.
To see this conceptual answer at work in a concrete example, let us take $\mathcal{C} := Top$ with the usual W (the weak homotopy equivalences). Take $I := (\bullet \longleftarrow \bullet \longrightarrow \bullet)$ (so that usual colimits along I are pushouts), and consider the following two I-diagrams in $\mathcal{C}$ (where the maps are the obvious ones):

$$pt \longleftarrow S^n \longrightarrow pt$$

$$D^n \longleftarrow S^n \longrightarrow D^n.$$

There is an obvious map of diagrams from the second one to the first one: the identity in the middle term and the shrinking map on the left and right terms. Note that this map of diagrams is "in" W, i.e.

each map is a weak homotopy equivalence. So the two diagrams should describe isomorphic elements in $W_I^{-1}(\mathcal{C}^I)$. But if we take the corresponding usual pushouts in Top, we get pt for the first diagram and S^{n+1} for the second diagram: the results are therefore not isomorphic in $\mathrm{Ho}(Top)$. This shows that usual colimits are invariant under weak equivalences, i.e. are not the right objects to consider when studying pairs $(\mathcal{C}, W)$ with an eye to $W^{-1}\mathcal{C}$.

Theorem 2.4.2 *If a pair $(\mathcal{C}, W)$ is part of a model structure, then $holim_I$ and $hocolim_I$ exist for any $I \in Cat$.*

If, for a given pair $(\mathcal{C}, W)$, one has $holim_I$ and $hocolim_I$, for any I (e.g. for pairs coming from model categories) one can look for mapping spaces as follows:

- for a simplicial set K and an object $y \in \mathcal{C}$, consider

$$y^K := holim_\Delta(\Delta \longrightarrow \mathcal{C} : [n] \longmapsto \prod_{K_n} y)$$

  as an object in $W^{-1}\mathcal{C}$; suppose that this gives a functor $\mathrm{Ho}(SSet) \longrightarrow W^{-1}\mathcal{C} : K \longmapsto y^K$ (i.e. that $K \simeq K'$ in $\mathrm{Ho}(SSet)$ yields $y^K \simeq y^{K'}$ in $W^{-1}\mathcal{C}$);

- Then, for $x, y \in \mathcal{C}$, we say that the mapping space relative to the pair $(\mathcal{C}, W)$ between x and y exists if the functor

$$\underline{Map}_{(\mathcal{C},W)}(x,y) : \mathrm{Ho}(SSet) \longrightarrow Sets : K \longmapsto [x, y^K] \equiv Hom_{W^{-1}\mathcal{C}}(x, y^K)$$

  is representable. The corresponding representative object in $\mathrm{Ho}(SSet)$ will be denoted by $Map_{(\mathcal{C},W)}(x,y)$ and called the *mapping space* relative to the pair $(\mathcal{C}, W)$ between x and y.

Theorem 2.4.3 *If a pair $(\mathcal{C}, W)$ is part of a model structure, then mapping spaces $Map_{(\mathcal{C},W)}(x,y)$ exist for any $x, y \in \mathcal{C}$.*

In other words, any model category is naturally enriched over $\mathrm{Ho}(SSet)$.

This was the *functorial* (or "*Derivateurs*") *approach* to mapping spaces of pairs. There is also a *resolution approach* to mapping spaces in a model category M (essentially due to Dwyer-Kan, see [Hi]) where mapping spaces are defined using left/right cosimplicial/simplicial resolutions of objects in M. The idea is as follows:

- There is a model structure on the category cM of cosimplicial objects in M (i.e. the diagram category M^Δ), called the Reedy model structure ([Hi]), in whch weak equivalences are defined objectwise. A *cosimplicial resolution* of an object $x \in M$ is then a weak equivalence $\Gamma(x)^* \longrightarrow c^*(x)$ in cM (where $c^*(x)$ is the constant cosimplicial object at x) with $\Gamma(x)^*$ Reedy cofibrant. dually, one defines *simplicial resolutions*.

- Functorial choices of cosimplicial/simplicial resolutions alwways exists in a model category (and their categories are contractible); moreover, if $\Gamma(-)^* \longrightarrow c^*(-)$ and $c_*(-) \longrightarrow \Sigma(-)_*$ are, respectively, functorial cosimplicial and simplicial resolutions in M, we have an isomorphism in $\mathrm{Ho}(SSet)$

$$diag(Hom_M(\Gamma(x)^*, \Sigma(y)_*)) \simeq Map_M(x,y)$$

  for any $x, y \in M$. Note also that if x is cofibrant (resp. if y is fibrant), we have $Hom_M(x, \Sigma(y)_*) \simeq Map_M(x,y)$ (resp. $Hom_M(\Gamma(x)^*, y) \simeq Map_M(x,y)$).

A particular nice case where the mapping spaces can be easily computed is when M is a *simplicial model category*. This essentially means that

- for any $x, y \in M$ there exists a simplicial set $\underline{Hom}_M(x,y)$ and a composition $\underline{Hom}_M(x,y) \times \underline{Hom}_M(y,z) \longrightarrow \underline{Hom}_M(x,z)$;
- there is an isomorphism $\underline{Hom}_M(-,-)_0 \simeq Hom_M(-,-)$ satisfying various compatibilities;
- for any $K \in SSet$ and any $x \in M$, there are functorially defined objects x^K and $x \otimes K$ in M such that
$$\underline{Hom}_M(x \otimes K, y) \simeq \underline{Hom}_{SSet}(K, \underline{Hom}_M(x,y)) \simeq \underline{Hom}_M(x, y^K).$$
- if $i : x \longrightarrow x'$ is a cofibration and $p : y \longrightarrow y'$ is a fibration in M, then the induced map
$$\underline{Hom}_M(x', y) \longrightarrow \underline{Hom}_M(y, x) \times_{\underline{Hom}_M(x,y')} \underline{Hom}_M(x', y')$$
is a fibration in $SSet$ which is moreover trivial if either i or p are trivial.

For a simplicial model category M, one defines
$$\mathbb{R}\underline{Hom}_M(x,y) := \underline{Hom}_M(Qx, Ry).$$
In such an M there is a canonical cosimplicial resolution functor $\Gamma(x)^* := Qx \otimes \Delta[n]$ and, for any $x, y \in M$, we have isomorphisms in $\mathrm{Ho}(Sset)$
$$Map_M(x,y) \simeq Map_M(Qx, Ry) \simeq Hom_M(Qx \otimes \Delta[*], Ry) \simeq Hom_{SSet}(\Delta[*], \underline{Hom}_M(Qx, Ry)) \simeq \mathbb{R}\underline{Hom}_M(x,y).$$

Remark 2.4.4 *Yet another approach to mapping spaces: the Dwyer-Kan localization* ([D-K1, D-K2, D-K3, D-K4, D-K5]). This is perhaps the most general and conceptual approach to mapping spaces for pairs $(\mathcal{C}, W)$. Let us come back to pairs $(\mathcal{C}, W)$: we have seen that the standard localization $W^{-1}\mathcal{C}$ does not contain all the higher homotopy informations carried by the pair itself. However, Dwyer and Kan have defined a richer notion of localization, called *simplicial* or *hammock localization* $L(\mathcal{C}; W)$. It is a simplicially enriched category with the same objects as $\mathcal{C}$ such that
$$\pi_0(\underline{Hom}_{L(\mathcal{C};W)}(x,y)) \simeq Hom_{W^{-1}\mathcal{C}}(x,y),$$
in other words the 1-truncation of $L(\mathcal{C}; W)$ gives back the usual localization $W^{-1}\mathcal{C}$. However, $L(\mathcal{C}; W)$ contains *all* the higher homotopical informations stored in the pair $(\mathcal{C}, W)$: for example, if M is a model category, then
$$Map_M(x,y) \simeq \underline{Hom}_{L(\mathcal{C};W)}(x,y).$$
So one can actually *define* mapping spaces for an arbitrary pair $(\mathcal{C}, W)$ as
$$Map_{(\mathcal{C},W)}(-,-) := \underline{Hom}_{L(\mathcal{C};W)}(-,-).$$
We like to think of $L(\mathcal{C}; W)$ as the *correct* ∞*-localization* of $\mathcal{C}$ with respect to W: in fact $L(\mathcal{C}; W)$ satisfies a higher categorical version of the universal property satisfied by the usual localization.

2.5 Morphisms between categories with equivalences and between model categories

The obvious notion of morphism between pairs (i.e. categories with equivalences) $(\mathcal{C}, W) \longrightarrow (\mathcal{C}', W')$ is that of a functor $f : \mathcal{C} \longrightarrow \mathcal{C}'$ such that $f(W) \subset W'$; this induces a functor on localizations $\mathrm{Ho}(f) : W^{-1}\mathcal{C} \longrightarrow W'^{-1}\mathcal{C}'$.
However, if $(\mathcal{C}, W)$ and $(\mathcal{C}', W')$ are parts of model structures, then there are more functors $\mathcal{C} \longrightarrow \mathcal{C}'$ entitled to "induce" functors on the associated homotopy categories. This is already a familiar feature for derived functors between derived (or Waldhausen) categories: as soon as a functor preserves resolutions of a certain kind (projective/injective/flat/flasque etc.) and quasi-isomorphisms between them, one can derive it. For model categories one has:

- *Left derived functors.* If $f : M \longrightarrow N$ is a functor between model categories that sends weak equivalences between cofibrant objects to weak equivalences, the restriction $f_c : (M_c, W_{M,c}) \longrightarrow (N, W_N)$ is a morphism of pairs as defined above; therefore there ia an induced left derived functor
$$\mathbb{L}f : \mathrm{Ho}(M) \xrightarrow{Q} \mathrm{Ho}(M_c) \xrightarrow{\mathrm{Ho}(f_c)} \mathrm{Ho}(N) .$$
- *Right derived functors.* If $g : M \longrightarrow N$ is a functor between model categories that sends weak equivalences between fibrant objects to weak equivalences, the restriction $g_f : (M_f, W_{M,f}) \longrightarrow (N, W_N)$ is a morphism of pairs as defined above; therefore there ia an induced right derived functor
$$\mathbb{R}g : \mathrm{Ho}(M) \xrightarrow{R} \mathrm{Ho}(M_f) \xrightarrow{\mathrm{Ho}(g_f)} \mathrm{Ho}(N) .$$
- *Mixed derived functors.* There are similar constructions when a functor $M \longrightarrow N$ sends weak equivalences between cofibrant and fibrant objects to weak equivalences (one get functors that may be denoted by $\mathbb{RL}f$ and $\mathbb{LR}f$).

Particular kind of functors as above are the so-called *left Quillen functors* (resp. *right Quillen functors*) that are righta adjoint functors preserving cofibrations and trivial cofibrations (resp. left adjoint functors preserving fibrations and trivial fibrations). In an adjunction (ordered as (left,right) adjoints) (f, g), f is left Quillen iff g is right Quillen; these adjunctions are often called Quillen pairs or Quillen adjunctions and deserve to be called *morphisms between model categories* (but one has to choose one of the two possible directions for the morphism), see [Ho, §1.4].
What about *isomorphisms* or better *equivalences* between model categories? First of all, note that if
$$f : M \rightleftarrows N : g$$
is a morphism between model categories (i.e. a Quillen pair where,say, f is the left adjoint), then
$$\mathbb{L}f : \mathrm{Ho}(M) \rightleftarrows \mathrm{Ho}(N) : \mathbb{R}g$$
is again an adjunction, called the *derived adjunction* of the given morphsim. Then, we say that a morphism of model categories is a *Quillen equivalence* if its derived adjunction is an equivalence (i.e. $\mathbb{R}g$ is quasi inverse to $\mathbb{L}f$).

Example 2.5.1 The adjunction
$$|-| : SSet \rightleftarrows Top : Sing$$
is a Quillen equivalence.

2.6 Homotopical localization

The reference for this subsection is [Hi].
We have already seen analogs of classical constructions in category theory (limits, colimits, morphisms between categories,...) for model categories; it turns out that there is also a suitable analog of the notion of localization.
Given a model category M and a subse S of maps in M, we formulate the following natural *homotopy localization problem*:

- Does it exist a *universal* model category M_S in which the maps in S are weak equivalences? In other words, can we formally add inverses to S in $\mathrm{Ho}(M_S)$ (not in M_S itself: that would be a usual localization)?

More precisely, like in the case of the usual localization, we want to consider morphisms $M \longrightarrow N$ of model categories such that... but which morphisms? One has left or right Quillen functors: this will in fact give rise to a left and a right homotopical localization. Let us take the "left" approach, to fix ideas. Therefore, we will consider left Quillen functors $f : M \longrightarrow N$ that inverts maps belonging to S *in* $\mathrm{Ho}(N)$. This is easily seen to be equivalent to requiring that the left derived functors $\mathbb{L}f : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(N)$ take (images in $\mathrm{Ho}(M)$ of) maps in S to isomorphisms. A *left homotopical localization* of M with respect to S will then be a Quillen functor with this property, which is initial.
Let us call an object $x \in M$ *S-local* if it "sees" maps in S as weak equivalences i.e. if it is fibrant and for any $y \longrightarrow y'$ in S, the induced map $Map_M(y', x) \longrightarrow Map_M(y, x)$ is an isomorphism in $\mathrm{Ho}(SSet)$.
A map $f : x \longrightarrow x'$ in M is an *S-local equivalence* if it is "seen" as an equivalence by any S-local object, i.e. if for any S-local object $y \in M$, the induced map $Map_M(x', y) \longrightarrow Map_M(x, y)$ is an isomorphism in $\mathrm{Ho}(SSet)$.

Theorem 2.6.1 (Bousfield-Hirschhorn) *If M is a "nice" model category (left proper and cellular, for the experts), then:*

1. *the following classes of maps in M*
   - *W_S:= S-local equivalences;*
   - Cof_S*:= cofibrations in M;*
   - Fib_S*:= maps with the right lifting property with respect to S-local equivalences which are also cofibrations in M,*

   are part of a model structure on M, denoted by $L_S M$;
2. *The identity functor on M induces a left Quillen functor $M \longrightarrow L_S M$ which is a left homotopical localization of M with respect to S.*

The category $L_S M$ is most commonly called the *left Bousfield localization* of M with respect to S. We conclude by listing, for future reference in the next lectures, the most basic properties of $L_S M$ (we denote by (W, Fib, Cof) the model structure on M):

- $W \subset W_S$;
- Trivial fibrations in M coincide with trivial fibrations in $L_S M$;
- $\mathrm{Fib}_S \subset \mathrm{Fib}$;
- Trivial cofibrations in M are trivial cofibrations in $L_S M$;
- Fibrant objects in $L_S M$ are exactly S-local objects;

- The right derived functor $\mathbb{R}\mathrm{Id}_M : \mathrm{Ho}(L_S M) \longrightarrow \mathrm{Ho}(M)$ is fully faithful and its essential image is the subcategory of $\mathrm{Ho}(M)$ consisting of objects $x \in M$ such that, for any $y \longrightarrow y'$ in S, the induced map $Map_M(y', x) \longrightarrow Map_M(y, x)$ is an isomorphism in $\mathrm{Ho}(SSet)$ (this is half of the condition for x being S-local: exactly, the only half that is invariant under weak equivalences);

- If M is a simplicial model category, then $L_S M$ is a simplicial model category (with the same tensored/cotensored/enriched structure).

3 Lecture 4. Stacks as simplicial presheaves.

3.1 The model category of simplicial presheaves on a Grothendieck site

For an algebraic geometer, usually, a stack is a category fibered in groupoids over some site of schemes and satisfying an additional descent, sheaf-like condition. However there are other ways to look at stacks: as a particular kind of simplicial presheaves (over the same site) or as particular lax 2-functors from the given site to Cat.
In all our talks, we will be considering stacks as simplicial presheaves on a given (sometimes "non-classical") site; so it is worthwile spending some time in explaining the relation between this point of view and the point of view (often denoted as "classical" in the following) of categories fibered in groupoids.
We will be somewhat brief and omit most of the proofs, since a full treatment of the subject would require a full DFG Schwerpunkt on its own. Details about most of the contents of the first two subsections, can be found in [Ja1, Ja2], [Hol] and [DHI]. Of course, our take here is massively influenced by the work of Carlos Simpson, in particular by [S1, S2, H-S].

3.1.1 The global model structure on simplicial presheaves

Before explaining the relations with the classical point of view, let us first introduce the point of view of simplicial presheaves on a Grothendieck site.

Let $(\mathcal{C}, \tau)$ be a Grothendieck site (that we will suppose to be small, just to avoid entering in multiple-universes choices). We will denote by $\mathrm{SPr}(\mathcal{C})$ the category of simplicial presheaves on $\mathcal{C}$, i.e. the category of controvariant functors $\mathcal{C}^{op} \to \mathrm{SSet}$. We already saw that the category SSet is a model category; this model structure induces a natural model structure on $\mathrm{SPr}(\mathcal{C})$.

Proposition 3.1.1 *The category* $\mathrm{SPr}(\mathcal{C})$ *endowed with the set of objectwise fibrations (resp. of objectwise equivalences) is a model category. This model structure will be called the* global projective *model structure on* $\mathrm{SPr}(\mathcal{C})$ *and denoted by* $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$.

Proof. Exercise (use the small object argument [Ho, Thm. 2.1.14]). □

An immediate remark: the projective model structure obviously does not *see* the topology on $\mathcal{C}$. A consequent question: can we modify this model structure so as to take into account, in some meaningful way, the given topology τ on $\mathcal{C}$?
Of course, one possible answer is to replace the category $\mathrm{SPr}(\mathcal{C})$ with the category $\mathrm{SSh}(\mathcal{C}, \tau)$ of simplicial *sheaves* on $(\mathcal{C}, \tau)$ and try to build a model structure in it. This can actually be done ([Jo1]), but here we prefer to keep working with simplicial presheaves: one can show that the two approaches are equivalent in some sense. It turns out that there is a standard way to "homotopically" invert all the coverings in the topology, considered as maps in $\mathrm{SPr}(\mathcal{C})$. This technique, already briefly discussed in Lecture 3 and called the left Bousfield localization gives a new model structure whose fibrant objects

have descent with respect to the topology. We will describe the main properties of this *local* model structure in the next subsection.

Let us close this subsection by recalling that $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$ is actually a *simplicial model category*([Hi, §10.1] or Lecture 3) whose simplicial enrichment is given by

$$\underline{Hom}(F,G)_n := Hom(F \times \Delta[n], G).$$

Therefore, for any $F, G \in \mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$ one has an isomorphism in $\mathrm{Ho}(SSet)$, $\mathrm{Map}_{\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}}(F,G) \simeq \mathbb{R}\underline{Hom}(F,G)$, where, as usual, Map_M denotes the mapping spaces in the model category M.

3.1.2 The local model structure on simplicial presheaves. Hyperdescent

We won't present the details of the Bousfield localization technique here but only give the most useful properties of it.

Given a morphism of simplicial presheaves $f : F \to G$ on $\mathcal{C}$, we say that f is a *τ-local equivalence* if it induces a weak equivalence of simplicial sets $f_x : F_x \to G_x$ on the stalks, for any point x in the site. Recall ([SGA4-I]) that a point in a site is just a point in the associated topos of sheaves of sets on the site, i.e. a geometric morphism x from the topos Set of sheaves of sets over a category with one object and one morphism, to our topos (so that we have an inverse image functor $x^* : \mathrm{Sh}(\mathcal{C}, \tau) \to \mathrm{Set}$ which is a left adjoint and left exact). The stalk at the point x of a simplicial presheaves F is then obtained from the levelwise composition of the sheafication functor followed by x^*.
Actually this definition is only correct if the site has enough points, property that we will suppose to make definitions easier. In the more general case, a local equivalence will be a map inducing an isomorphism on all the *sheaves of homotopy groups* of F and G, for any base point.

For any covering family $(U_i \to X)$, we may consider the corresponding (Čech) nerve $N(U)_\bullet$ which is the simplicial object in $\mathcal{C}$ defined by

$$N(U)_n := \prod U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n}.$$

Note that there is a natural augmentation $N(U)_\bullet \to X$. This is only a special case of a more general kind of simplicial object in $\mathcal{C}$ augmented over X, called *τ-hypercover* of X. An hypercover is essentially a Cech nerve in which we allow ourselves to refine each stage by taking some further covering in the given topology. We won't need the general definition of hypercovering, at least at this point.

Definition 3.1.2 *We say that a simplicial presheaf F has* τ-hyperdescent *if for any object $X \in \mathcal{C}$ and any τ-hypercovering $U_\bullet \to X$ the canonical map in* Ho(SSet)

$$F(X) \longrightarrow \mathrm{holim} F(U_\bullet)$$

(where on the r.h.s. we wrote $F(U_\bullet)$ for $Hom(U_\bullet, F)$, following the common Yoneda-abuse)is an isomorphism.

Note that if F is the constant simplicial presheaf induced by a presheaf $\mathcal{F}$ of sets on $(\mathcal{C}, \tau)$, and $N(U_\bullet) \to X$ is the Cech nerve of some covering $(U_i \to X)$, then F has hyperdescent with respect to $U_\bullet$ iff $\mathcal{F}$ has the usual sheaf property with respect to the covering $(U_i \to X)$. In fact, for constant simplicial sets holim = lim and weak equivalences between constant simplicial sets are just set isomorphisms. So hyperdescent is really a homotopical generalization of the usual descent or sheaf property.

For any covering family $(U_i \to X)$, we view it as a set of maps in $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$ and pass to the corresponding Cech nerve $N(U)_\bullet$ which is now a simplicial object in $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$. There is an augmentation $N(U)_\bullet \to X$ and we consider its homotopy colimit $\mathrm{hocolim}(N(U)_\bullet) \to X$ computed in the global model structure. We can perform the same procedure for any *τ-hypercovering* $U_\bullet \to X$; this way we get a (huge) set of maps $\mathrm{hc}(\tau)$ in $\mathrm{Ho}(\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}})$. We denote by $\mathrm{HC}(\tau)$ the set of all maps in $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$ which projects into $\mathrm{hc}(\tau)$ via the localization functor.

Theorem 3.1.3 *There exists a model structure* $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ *on the category* $\mathrm{SPr}(\mathcal{C})$*, called the local model structure, in which:*

1. *the equivalences are exactly the local equivalences;*
2. *the identity functor* $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}} \longrightarrow \mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ *is a left Quillen functor and its left derived functor* $\mathbb{R}\mathrm{Id} : \mathrm{Ho}(\mathrm{SPr}(\mathcal{C})_{\mathrm{loc}}) \longrightarrow \mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{glob}})$ *is fully faithful with essential image the full subcategory of simplicial presheaves having* hyperdescent *with respect to* τ*.*

The proof is in two steps. *First* one defines $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ as the left Bousfield localization of $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$ with respect to the set of maps $\mathrm{HC}(\tau)$[2]. By general properties of the Bousfield localization machinery, this already identifies local fibrant objects as those global fibrant objects that have *tau* -hyperdescent and this establishes the second part of Theorem 3.1.3. *Then* one need a bit of work to proves that the equivalences in Bousfield localized model structure (which are the so-called $\mathrm{HC}(\tau)$*-local equivalences*) actually coincides with the local equivalences previously defined. For a complete proof, the reader may wish to consult [DHI].

3.1.3 Hom stacks

The category $\mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}})$ of ∞-stacks has a very rich structure. First, since it is the homotopy category of a simplicially enriched model category (where $\underline{Hom}(F,G)_n := Hom(F \times \Delta[n], G)$), it is *enriched over* $\mathrm{Ho}(\mathrm{SSet})$, by defining $\mathbb{R}\underline{Hom}(F,G) = \underline{Hom}(QF, RG)$. Moreover, it is a cartesian closed category i.e it has *internal Hom-objects* (just like the category of sheaves over a topological space): for any F and G in $\mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}})$, one has an object $\mathbb{R}\mathcal{H}om(F,G)$ in $\mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}})$ such that

$$Hom(F', \mathbb{R}\mathcal{H}om(F,G)) \simeq Hom(F' \times F, G).$$

Of course, this essentially comes from the fact that the category of simplicial presheaves on a site has itself internal Hom-objects defined as usual

$$\mathcal{H}om(F,G) : X \longmapsto Hom_{\mathrm{SPr}(\mathcal{C}/X)}(F_{|\mathcal{C}/X}, G_{|\mathcal{C}/X}).$$

Remark 3.1.4 We are a bit cheating here, since the construction requires a small detour. If $\mathcal{C}$ has fibered products, then one simply define

$$\mathbb{R}\mathcal{H}om(F,G) := a(\mathcal{H}om(QF, RG)),$$

where $a = \mathbb{L}\mathrm{Id} : \mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{glob}} \to \mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ denotes the stackification functor and R (resp. Q) is a fibrant (resp. cofibrant) replacement in the projective local model structure. Here is an overview of the construction in the general case (for more details, see [HAG-I]). First consider the injective model structure on $\mathrm{SPr}(\mathcal{C})$ in which equivalences and *cofibrations* are defined objectwise. Then, for any F and G in $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ define the corresponding internal Hom-stack as

$$\mathbb{R}\mathcal{H}om(F,G) := a(\mathcal{H}om(F, R_{\mathrm{inj}}G)),$$

where $a = \mathbb{L}\mathrm{Id} : \mathrm{SPr}(\mathcal{C})_{\mathrm{glob}} \to \mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ denotes the stackification functor and R_{inj} the fibrant replacement functor in the injective model structure.

[2] Actaully $\mathrm{HC}(\tau)$ is not a set so we need to find a “nice” set in it and to left Bousfield localize with respect to this: see [DHI] or [HAG-I].

3.2 Reinterpreting sheaves and stacks in groupoids as truncated simplicial presheaves

We keep the previous notations: $(\mathcal{C},\tau)$ is the base site and $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ is the local model structure on the category $\mathrm{SPr}(\mathcal{C})$.
Let us denote by $\mathrm{St}(\mathcal{C},\tau)$ the category of stacks fibered in groupoids over the site $(\mathcal{C},\tau)$ ([La-Mo]); its objects are stacks $\mathcal{S}\to\mathcal{C}$ and the morphisms between $\mathcal{S}\to\mathcal{C}$ and $\mathcal{S}'\to\mathcal{C}$ are the obvious (strictly) commutative triangles of functors. Note that one may consider $\mathrm{St}(\mathcal{C},\tau)$ also as a 2-category (in the obvious way, laxifying commutative triangles); it will be denoted by $\mathbf{St}(\mathcal{C},\tau)$.

3.2.1 Stacks in groupoids as 1-truncated simplicial presheaves

We want to associate to any stack in groupoids over $(\mathcal{C},\tau)$ a simplicial presheaf on $\mathcal{C}$. If $\mathcal{S}\to\mathcal{C}$ is a prestack or a stack in groupoids, the rule $C\mapsto\mathcal{S}_C$ is not "exactly" a presheaf of groupoids on $\mathcal{C}$. In fact any inverse image functor $(-)^*$ only satisfy the weak transitivity $(f\circ g)^*\simeq g^*\circ f^*$, so that the rule $C\mapsto\mathcal{S}_C$ only defines a priori a lax (or weak or pseudo) presheaf of groupoids. However, one can always associate to such an $\mathcal{S}\to\mathcal{C}$ a genuine presheaf of groupoids and actually of simplicial sets, on $\mathcal{C}$ through the following *strictification* (or *canonical clivage*) construction.
Recall that for groupoids, the nerve functor is often called the *classifying space* functor and denoted by B; the reason for this name is that for a groupoid $\mathcal{G}$, the fundamental groupoid $\Pi_1(\mathrm{Nerve}(\mathcal{G}))$ is equivalent to $\mathcal{G}$ and $\mathrm{Nerve}(\mathcal{G})$ has trivial higher homotopy groups: i.e. it behaves analogously to the classifying space of a group.

Proposition 3.2.1 (Strictification for prestacks in groupoids.) *Let $\mathcal{S}\to\mathcal{C}$ be a prestack in groupoids over $(\mathcal{C},\tau)$. The rule*

$$B\mathcal{S}: C\longmapsto \mathrm{B}\underline{Hom}_{\mathrm{FibGrpd}/\mathcal{C}}(C,\mathcal{S})$$

defines a simplicial presheaf $B\mathcal{S}$ on $\mathcal{C}$. Here $\underline{Hom}_{\mathrm{Grpd}/\mathcal{C}}(-,-)$ denotes the groupoid of morphisms between categories fibered in groupoids over $\mathcal{C}$.
Moreover, the rule $\mathrm{B}:\mathcal{S}\mapsto\mathrm{B}\mathcal{S}$ defines a functor between the category of prestacks in groupoids over $(\mathcal{C},\tau)$ and the category of simplicial presheaves over $\mathcal{C}$.

Proof. Long, but safe check. □

Note that if $\mathcal{S}\to\mathcal{C}$ is a stack in groupoids, it satisfies by definition a descent condition on $(\mathcal{C},\tau)$, and this implies that $B\mathcal{S}$ also satisfies the hyperdescent condition in $\mathrm{SPr}(C)$ and actually is a fibrant object in $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ (the fact that descent with respect to only nerves of coverings implies full hyperdescent is due to the observation that B of a stack is a 1-truncated simplicial presheaf while the fact that not only B of a stack staisfies hyperdescent but is also objectwise, i.e. globally, fibrant follows from noticing that the classifying simplicial set of a groupoid is always a fibrant simplicial set).

Remark 3.2.2 On the other hand, if $F\in\mathrm{SPr}(C)$, we may first associate to it a presheaf in groupoids Π_1F which sends $X\in\mathcal{C}$ to $\Pi_1F(X)$; then we may apply the Grothendieck construction (explain?) to Π_1F to get a genuine pre-stack in groupoids $\int\Pi_1F$ on $(\mathcal{C},\tau)$. This is not in general a stack unless one starts from a simplicial presheaf F being fibrant in $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$ (actually it is enough that F satisfies restricted hyperdescent, i.e. hyperdescent with respect to nerves of all τ-coverings).

Using the nerve functor on the Hom groupoids, one can view the 2-category $\mathbf{St}(\mathcal{C},\tau)$ of stacks in groupoids as a category enriched over SSet; we will denote it by $\underline{\mathrm{St}(\mathcal{C},\tau)}$.

Exercise 3.2.3 Check that one can enhance the functor B to a simplicial functor $\underline{B} : \underline{\mathrm{St}(\mathcal{C},\tau)} \to \underline{\mathrm{SPr}_\tau(\mathcal{C})}$.

By composing this enhanced $\underline{B}$ with a simplicial cofibrant replacement functor $\underline{Q} : \underline{\mathrm{SPr}_\tau(\mathcal{C})} \to \underline{\mathrm{SPr}_\tau(\mathcal{C})^c_{\mathrm{loc}}}$ (it exists by ....) and noticing (again) that B of a stack is always a fibrant object in $\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}$, we get a simplicial functor

$$\underline{Q}\underline{B} : \underline{\mathrm{St}(\mathcal{C},\tau)} \longrightarrow \underline{\mathrm{SPr}_\tau(\mathcal{C})^{cf}_{\mathrm{loc}}}.$$

We are now ready to state the comparison between stacks in groupoids and simplicial presheaves

Theorem 3.2.4 *The 2-truncation*

$$(\underline{Q}\underline{B})_{\leq 2} : \underline{\mathrm{St}(\mathcal{C},\tau)}_{\leq 2} \longrightarrow \underline{\mathrm{SPr}_\tau(\mathcal{C})^{cf}_{\mathrm{loc}}}_{\leq 2}$$

is a fully faithful morphism of 2-categories (i.e. it induces an equivalence of categories between the Hom-categories) and its essential image is the full 2-subcategory of $\underline{\mathrm{SPr}_\tau(\mathcal{C})^{cf}_{\mathrm{loc}}}_{\leq 2}$ *consisting of 1-truncated simplicial presheaves.*

Since $\Pi_1 \circ \mathrm{Nerve}$ is canonically isomorphic to the identity functor on Grpd, the 2-truncation 2-category $\underline{\mathrm{St}(\mathcal{C},\tau)}_{\leq 2}$ is canonically equivalent to the 2-category of stacks in groupoids. Therefore, Theorem 3.2.4 says that *one can embed the theory of stacks in groupoids in the theory of simplicial presheaves* on the same site.

Corollary 3.2.5 *The 1-truncation of* ($\underline{Q}\underline{B}$ *defines a fully faithful functor*

$$(1-\mathrm{iso})^{-1}(\mathrm{St}(\mathcal{C},\tau)) \hookrightarrow \mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}})$$

whose essential image is given by the full subcategory of 1-truncated simplicial presheaves.

Remark 3.2.6 Actually Theorem 3.2.4 is equivalent to the following statement: $\underline{Q}\underline{B}$ induces a fully faithful morphism between the Dwyer-Kan *simplicial localizations*

$$\mathrm{L}(\mathrm{St}(\mathcal{C},\tau), 1-\mathrm{iso}) \longrightarrow \mathrm{L}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}, \mathrm{equiv}).$$

The reason for the equivalence of the two statements is that the simplicial category $\mathrm{L}(\underline{\mathrm{St}(\mathcal{C},\tau)}, 1-\mathrm{iso})$ is 2-truncated (i.e. its simplicial Hom's are 1-truncated) and this follows from the fact that it is a localization of a model category with an enrichment in groupoids, whose mapping spaces are exactly computed by the groupoid of morphims (which are 1-truncated when viewed as simplicial sets).

3.2.2 Sheaves of sets as 0-truncated simplicial presheaves

Let $\mathrm{Pr}(\mathcal{C})$ be the category of presheaves of sets on the site $(\mathcal{C},\tau)$. The notion of τ-covering induces a natural notion of *τ-local isomorphsim* of presheaves: a map of presheaves is a local isomorphism if it is surjective and injective "up to a covering refinement" (not difficult to figure out a precise definition: exercise). Then we may localize the category $\mathrm{Pr}(\mathcal{C})$ with respect to the set W_τ of local isomorphisms (i.e., we formally invert them); the category $W_\tau^{-1}\mathrm{Pr}(\mathcal{C})$ we obtain, comes naturally endowed with a localization functor $\mathrm{loc} : \mathrm{Pr}(\mathcal{C}) \to W_\tau^{-1}\mathrm{Pr}(\mathcal{C})$ which can be checked to be left exact (i.e. commuting with finite limits) and to have a right adjoint. We will call a localization with these properties a *left exact localization*. The crucial property of this construction is that $W_\tau^{-1}\mathrm{Pr}(\mathcal{C})$ is naturally equivalent

to the category of *sheaves* of sets on $(\mathcal{C},\tau)$ and through this equivalence, the localization functor loc is identified with the sheafication functor.
The properties of the left Bousfield localization (and Theorem 3.1.3) show that if we replace the category of presheaves with the category of simplicial presheaves, the τ-local isomorphisms by the τ-local equivalences and the usual localization by the its homotopy analog, the left Bousfield localization, the identity functor induces a *homotopy left exact localization* $\mathbb{L}\mathrm{Id} : \mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{glob}}) \to \mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}})$, i.e. $\mathbb{L}\mathrm{Id}$ has a right adjoint and is homotopy left exact (i.e. commutes with homotopy fibred products). These gives a further analogy (other than the hyperdescent property, Theorem 3.1.3) with the case of sheaves of sets, and it suggests again that objects in $\mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}})$ can be considered as homotopy analogs of sheaves on the given site. We will call these objects ∞-*stacks*.
Moreover, one can actually see usual sheaves of sets as ∞-stacks, in much the same way as we have seen stacks in groupoids as ∞-stacks in the previous subsection. The idea is the same and actually, technically simpler here: we may view any (pre)sheaf of sets as a constant simplicial presheaf. Composing with the localization to the homotopy category, this gives a functor

$$i : \mathrm{Sh}(\mathcal{C},\tau) \to \mathrm{Ho}(\mathrm{SPr}_\tau(\mathcal{C})_{\mathrm{loc}}).$$

Since any constant simplicial set has vanishing homotopy groups in dimensions ≥ 1, it is clear that the image of i consists of 0-truncated simplicial presheaves.

Proposition 3.2.7 *The functor i is fully faithful and its essential image consists of 0-truncated simplicial presheaves.*

Therefore, also the theory of sheaves of sets on the site $(\mathcal{C},\tau)$ is embedded in the theory of ∞-stacks over $(\mathcal{C},\tau)$.

3.3 Geometric stacks. Examples

3.3.1 Geometric stacks

In this section we specialize our base site to some site of schemes; to fix ideas, we will take $(\mathcal{C},\tau) := (\mathrm{Aff/R}, \text{ét})$, the big étale affine site over $\mathrm{Spec}R$. The treatment here is strongly influenced by [S2]. For such sites, in wich one has a notion of *affine object* and of *smooth morphism*, it makes sense to single out a special subcategory of ∞-stacks, called *geometric* stacks (or algebraic, in the case of usual stacks in groupoids, [La-Mo, Def. 4.1]). These are the stacks on which one can really hope to do some geometry (hence the name) in much the same way as one can work on smooth manifolds by knowing that they admit a smooth atlas. We will say that a stack in groupoids is *geometric* if it has a representable affine diagonal and a smooth algebraic space atlas i.e. a smooth surjective map from an algebraic space; this is slightly stronger than saying that it is algebraic in the sense of [La-Mo, Def. 4.1], in that in an algebraic stack one only requires a representable diagonal which is moreover separated and quasi-compact.

Definition 3.3.1 *Let F and G be ∞-stacks on* $(\mathrm{Aff/R}, \acute{e}t)$ *(i.e. objects in* $\mathrm{Ho}(\mathrm{SPr}_{\acute{e}t}(\mathrm{Aff/R})_{\mathrm{loc}})$*). We say that*

1. *a morphism $F \to G$ in* $\mathrm{Ho}(\mathrm{SPr}_{\acute{e}t}(\mathrm{Aff/R})_{\mathrm{loc}})$ *is a* covering *if it induces an epimorphism on the sheaves associated to the presheaves $X \mapsto \pi_0(F(X))$ and $X \mapsto \pi_0(G(X))$ of connected components of F and G.*
2. *a morphism $F \to G$ in* $\mathrm{Ho}(\mathrm{SPr}_{\acute{e}t}(\mathrm{Aff/R})_{\mathrm{loc}})$ *is* affine *(resp.,* affine smooth*) if for any commutative R-algebra A and any morphism* $\mathrm{Spec}A \to G$*, the corresponding homotopy base-change* $F \times^{\mathrm{h}}_G \mathrm{Spec}A$ *is isomorphic (in* $\mathrm{Ho}(\mathrm{SPr}_{\acute{e}t}(\mathrm{Aff/R})_{\mathrm{loc}})$*) to some* $\mathrm{Spec}B$*, for an R-algebra B (resp., and the morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *corresponds to a smooth morphism of affine schemes).*

With these definitions, we may imitate Artin's definition of an algebraic stack in groupoids and give the following

Definition 3.3.2 *An ∞-stack is* geometric *if it has an affine diagonal and admits a smooth atlas of affines, i.e. a smooth covering $\coprod_i \mathrm{Spec}A_i \to F$.*

Note that requiring that F has an affine diagonal is equivalent to requiring that any map $\mathrm{Spec}A \to F$ is affine (check as an exercise).

Exercise 3.3.3 Show that with this definition, geometric stack in groupoids (as defined above), when viewed as an ∞-stack is geometric (use that any algebraic space has an étale cover by a disjoint union of affine schemes).

Actually, also the converse of the previous exercise is true:

Proposition 3.3.4 *Any affine morphism of ∞-stacks induces an isomorphism on each π_i sheaf, for any $i \geq 2$ and a monomorphism for $i = 1$.*

Proof. Let $F \to G$ be affine. Pick any $\mathrm{Spec}A$-point in F and consider its image point in G. Then take the homotopy fiber in F at this point and consider the associated long exact sequence of sheaves of homotopy groups. Recalling that $\mathrm{Spec}A$ is 0-truncated yields the result. □

Corollary 3.3.5 *Any ∞-stack with an affine diagonal is 1-truncated. In particular, any geometric ∞-stack is isomorphic to (the image of) a geometric stack in groupoids.*

Proof. It is enough to apply the previous Proposition and to notice that the diagonal map induce again the diagonal map on sheaves of homotopy groups $\pi_i(F) \to \pi_i(F) \times \pi_i(F) \simeq \pi_i(F \times F)$. □

Thererore, as far as geometric ∞-stacks are concerned we do not really get new objects (other than those coming from geometric stacks in groupoids). Moreover note that, for example, schemes with non-affine diagonal are not geometric stacks. However, one can complicate further the "geometricity" of the stacks (i.e. enlarging the category of geometric stacks) by essentially upgrading geometric stacks to affines i.e. by replacing in the above definitions, "affine" with "geometric". Let us only give the next step. To make the inductive definition more transparent let us change the previous notations as follows.

- An affine scheme will be called a *0-representable* ∞-stack;
- An affine morphism between ∞-stacks will be called a *0-representable morphism*;
- a geometric ∞-stack will be called a *1-geometric* ∞-stack.

Definition 3.3.6 *Let F and G be ∞-stacks on $(\mathrm{Aff/R}, \acute{e}t)$. We say that*

1. *a morphism $F \to G$ in $\mathrm{Ho}(\mathrm{SPr}_{\acute{e}t}(\mathrm{Aff/R})_{\mathrm{loc}})$ is* 1-representable *if for any commutative R-algebra A and any morphism $\mathrm{Spec}A \to G$, the corresponding homotopy base-change $F \times^{\mathrm{h}}_G \mathrm{Spec}A$ is 1-geometric.*
2. *a 1-representable morphism $F \to G$ is* smooth *if, for any morphism $\mathrm{Spec}A \to G$, the morphism obtained by composing the base-cange map $F' \to \mathrm{Spec}(A)$ (where F' is 1-geometric by definition of 1-representable morphism) with a smooth atlas $\coprod_i \mathrm{Spec}B_i \to F'$ for F', comes from a smooth map of schemes (i.e. any $\mathrm{Spec}B_i \to \mathrm{Spec}A$ is smooth).*

Then, obviously

Definition 3.3.7 *An ∞-stack F is* 2-geometric *if its diagonal is 1-representable and it has a smooth atlas of 1-geometrics i.e. there exist 1-geometric stacks U_i and a 1-representable morphism $\coprod_i U_i \to F$ which is a smooth covering.*

Note that if F has a 1-geometric diagonal then any morphism $G \to F$, with G 1-geometric, is authomatically 1-representable. Moreover, in the definition above, we may assume that the atlas of 1-geometrics is actually an atlas of affines (i.e. that U is of the form $\coprod_i \mathrm{Spec}A_i$). We leave to the reader the exercise of setting up the inductive definition of n-representable morphism and of n-geometric ∞-stack. An ∞-stack will be called *geometric* it it is n-geometric for some n. The following exercise collects some of their properties, the most relevant of which is the last one that crucially depends from the fact that objects in our site (affine schemes in this case) are discrete objects (i.e constant simplicial presheaves) and in particular are 0-truncated.

Exercise 3.3.8

1. The (homotopy fiber) product of two n-geometric ∞-stacks is n-geometric.
2. For a morphism of ∞-stacks, the property of being n-representable is stable under homotopy base change.
3. If F is an ∞-stack and its diagonal is n-representable then any morphism to F from a n-representable ∞-stack is n-representable.
4. Any n-geometric ∞-stack is n-truncated (hint: use the long exact sequence of sheaves of homotopy groups to prove a result analogous to Proposition 3.3.4 and deduce the analog of Corollary 3.3.5).

3.3.2 The ∞-stack of perfect complexes

We give here a brief sketch of the construction of the ∞-stack of perfect complexes on the site $(\mathrm{Aff}/R, \text{ét})$.

For $A \in \mathrm{Aff}/R$, we denote by $\mathrm{Perf}(A)$ the category of perfect complexes over $\mathrm{Spec}A$, i.e. of complexes quasi-isomorphic to bounded complexes of vector bundles. Let us denote by $\mathrm{Perf}(A)^c$ the full subcategory of $\mathrm{Perf}(A)$ consisting of cofibrant complexes in the projective model structure on $\mathrm{Ch}(A)$ (see [Ho, §2.3]) and by $\mathrm{qiso}(\mathrm{Perf}(A)^c)$ the subcategory of quasi-isomorphisms in it. Since every object is cofibrant, the pull-back is well defined and we get a lax 2-functor

$$\widetilde{\mathrm{Perf}} : (\mathrm{Aff}/R)^{\mathrm{op}} \to \mathrm{Cat} : A \to \mathrm{qiso}(\mathrm{Perf}(A)^c).$$

Let us now apply the Grothendieck construction to $\widetilde{\mathrm{Perf}}$ to get a category $\int \widetilde{\mathrm{Perf}}$ fibered in categories over Aff/R. Recall that objects in $\int \widetilde{\mathrm{Perf}}$ are pairs $(A \in \mathrm{Aff}/R, a^\bullet \in \widetilde{\mathrm{Perf}}(A))$ and a morphism $(A, a^\bullet) \to (B, b^\bullet)$ is a pair (f, α) where $f : A \to B$ is a morphism in Aff/R and $\alpha : a^\bullet \to (f^*(b^\bullet))$ is a morphism in $\widetilde{\mathrm{Perf}}(A)$ (the composition is defined using the natural transformation $\lambda_{f,g} : g^* \circ f^* \to (f \circ g)^*$ which is part of the lax 2-functor data).
So we have a category $\int \widetilde{\mathrm{Perf}}$ fibered in categories over Aff/R. Now we strictify it to a genuine presheaf $\widehat{Perf}$ of categories over Aff/R, by defining

$$\widehat{Perf} : A \mapsto \mathrm{Hom}_{\mathrm{Cat}/(\mathrm{Aff}/R)}(A, \int \widetilde{\mathrm{Perf}}).$$

Finally, we compose $\widehat{Perf}$ with the nerve functor $\mathrm{Nerve} : \mathrm{Cat} \to \mathrm{SSet}$, to get a simplicial presheaf denoted by $\underline{\mathrm{Perf}}$. One can prove that this is actually a ∞-stack called the *∞-stack of perfect complexes* over the site $(\mathrm{Aff}/R, \text{ét})$. Moreover this stack is geometric, actually the union of an infinite number of finite-geometric stacks, the substacks corresponding to complexes of bounded universal cohomological amplitude. For example, the substack $\underline{\mathrm{Perf}}^{[0,0]}$ of complexes of cohomological amplitude 0 is the same as the stack $\underline{\mathrm{Vect}}$ of vector bundles over the site $(\mathrm{Aff}/R, \text{ét})$ and is therefore 1-geometric; the substack $\underline{\mathrm{Perf}}^{[0,1]}$ of complexes of cohomological amplitude 1 is a 2-geometric stack, etc. Note that $\underline{\mathrm{Perf}}$ is an absolute object, in the sense that for any scheme X over $\mathrm{Spec}R$, the *∞-stack of perfect complexes over X* is defined to be $\underline{\mathrm{Perf}}(X) := \mathcal{H}om(X, \underline{\mathrm{Perf}})$, where $\mathcal{H}om$ denotes the internal Hom-object in $\mathrm{Ho}(\mathrm{SPr}_{\text{ét}}(\mathrm{Aff}/R)_{\mathrm{loc}})$. Usually, geometricity statements about $\underline{\mathrm{Perf}}(X)$ are harder to establish; for example, it is probably true that if $R = k$ is a field and X is projective over R, then the substack $\underline{\mathrm{Perf}}^{[0,1]}(X)$ is again 2-geometric.

4 Lectures 5 and 6. Stacks over a model site

The aim of lectures 5-6 is to extend the theory of stacks (understood in the simplicial-presheaves' approach, as explained in Lecture 4) to *model sites* i.e. model categories endowed with a suitable homotopical modification of a Grothendieck topology.

4.1 Model category of prestacks on a model category

Recall that the first step in the construction of stacks on a Grothendieck site, was the construction of the model category of prestacks (denoted by $\mathrm{SPr}(\mathcal{C})_{\mathrm{glob}}$ in Lecture 4):

$$\mathcal{C} \rightsquigarrow \mathrm{SPr}(\mathcal{C}) : \textit{model category of prestacks on } \mathcal{C} \textit{ (projective model structure).}$$

Here $\mathcal{C}$ gives no non-trivial homotopical input (in other words it is a model category, a priori, only wit the trivial model structure).
Now, let (M, W) be a *model category*. We first perform the same construction, forgetting that M has a model structure:

$$M \rightsquigarrow \mathrm{SPr}(M) : \textit{projective model structure.}$$

Recall that this is a *simplicial model category*; in particular, it has

- a *$SSet$-tensored* structure: for $F \in \mathrm{SPr}(M)$ and $K \in SSet$, we define $F \otimes K \in \mathrm{SPr}(M)$ as $(F \otimes K)(x) := F(x) \times K$, for any $x \in M$;
- a *$SSet$-cotensored* structure: for $F \in \mathrm{SPr}(M)$ and $K \in SSet$, we define $F^K \in \mathrm{SPr}(M)$ as $(F^K)(x) := F(x)^K \equiv \underline{Hom}_{Sset}(K, F(x))$, for any $x \in M$;
- a *$SSet$-enrichment*: $\underline{Hom}(F, G)_n := Hom(F \otimes \Delta[n], G)$, for any $n \geq 0$, with the (generalized) adjunctions
$$\underline{Hom}(F \otimes K, G) \simeq \underline{Hom}(F, G^K) \simeq \underline{Hom}_{SSet}(K, \underline{Hom}(F, G)).$$

The model structure on $\mathrm{SPr}(M)$ does not "know" that M is a model category: this has the drawback that two Quillen-equivalent model categories may have non Quillen-equivalent $\mathrm{SPr}(-)$. Another drawback is that the usual Yoneda embedding $h_{-} : M \longrightarrow \mathrm{Pr}(M) \longrightarrow \mathrm{SPr}(M)$ does not preserve weak equivalences. Therefore we would like to modify the model structure on $\mathrm{SPr}(M)$ (keeping the same underlying category).
Natural idea: make weak equivalences in M weak equivalences in $\mathrm{SPr}(M)$.
Therefore, a way to remedy the previous drawbacks is to take the homotopy localization of $\mathrm{SPr}(M)$ with respect to equivalences in M.

Definition 4.1.1 *Let* $h_W := \{h_x \longrightarrow h_y | (x \longrightarrow y) \in W\}$. *The* model category of prestacks on M *is the left Bousfield localization of* $\mathrm{SPr}(M)$ *with respect to* h_W*:*

$$M^\wedge := L_{h_W}(\mathrm{SPr}(M)).$$

Here $h_x \in \mathrm{SPr}(M)$ is the value of the usual Yoneda embedding at $x \in M$

$$h_x(y) := c(Hom_M(y, x))$$

where $c : Set \longrightarrow SSet$ is the constant simplicial set functor.
By the properties of the left homotopical localization (see end of Lecture 3), we immediately deduce:

- Cofibrations in $M^\wedge$ are exactly the cofibrations in $\mathrm{SPr}(M)$.
- Fibrant objects in $M^\wedge$ are simplicial presheaves F on M such that
  1. F is objectwise fibrant;
  2. F preserves weak equivalences (Prove this as an exercise! Hint: use the explicit form of mapping spaces in the simplicial model category $\mathrm{SPr}(M)$ and the $SSet$-enriched Yoneda lemma).
- $\mathrm{Id} : \mathrm{SPr}(M) \longrightarrow M^\wedge$ is left Quillen; $\mathrm{Id} : M^\wedge \longrightarrow \mathrm{SPr}(M)$ is right Quillen.
- $\mathbb{R}(\mathrm{Id}) : \mathrm{Ho}(M^\wedge) \hookrightarrow \mathrm{Ho}(\mathrm{SPr}(M))$ is fully faithful and its essential image consists of those simplicial presheaves preserving equivalences.

4.2 Model Yoneda lemma

We would like to see objects in M as simplicial presheaves on M but in a homotopy invariant way.
The first idea is obviously to consider the usual Yoneda embedding

$$M \xrightarrow{h_-} \mathrm{Pr}(M) \xrightarrow{c} \mathrm{SPr}(M)$$

$$x \longmapsto h_x : y \longmapsto c(Hom_M(y, x))$$

Problem: This does not preserve equivalences (not even between (co)fibrant objects, hence cannot even derive it).
But, recall that in the previous section we have built $M^\wedge$ exactly by homotopy inverting equivalences in M i.e. maps $h_x \longrightarrow h_y$ where $(x \longrightarrow y) \in W$. Therefore the fully faithful functor

$$h_- : M \longrightarrow M^\wedge$$

preserves equivalences, hence yields a functor

$$\mathrm{Ho}(h) : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^\wedge).$$

Question: Is it still fully faithful?

Recalling that in M we have *mapping spaces*, there is also another natural candidate for our model Yoneda embedding.
Choose a cosimplicial resolution functor (see Lecture 3) $\Gamma(-)^* : M \longrightarrow cM = M^\Delta$ with a natural transformation

$$\Gamma(-)^* \longrightarrow c^*(-)$$

and define

$$\underline{h}_- : M \longrightarrow \mathrm{SPr}(M)$$

$$x \longmapsto Hom_M(\Gamma(-)^*, x).$$

Does this also induce a functor on the associated homotopy categories? Well, in general it does not preserve equivalences; however

1. it preserves fibrant objects (because ammping spaces are always fibrant simplicial sets);
2. it preserves equivalences between fibrant objects.

So it may be right-derived

$$\mathbb{R}h_- : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(\mathrm{SPr}(M)) : x \longmapsto \underline{h}_{R(x)}.$$

However, since $M^\wedge$ is the left Bousfield localization $\mathrm{SPr}(M)$ with respect to h_W, we easily see that (1) and (2) above also imply that

$$\underline{h}_- : M \longrightarrow M^\wedge$$

has the same properties (i.e. preserves fibrant objects and weak equivalences between them). Therefore, we also have

$$\mathbb{R}h_- : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^\wedge).$$

Question: Is this fully faithful?

Theorem 4.2.1 (Model Yoneda lemma)

- *The functors* $\mathrm{Ho}(h_-), \mathbb{R}\underline{h} : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^\wedge)$ *are canonically isomorphic, more precisely, the canonical map* $h_- \longrightarrow \underline{h}_{R(-)}$ *(induced by the natural transformation* $\Gamma(-)^* \longrightarrow c^*(-)$*) is a weak equivalence in* $M^\wedge$*.*
- $\mathrm{Ho}(h_-)$ *and* $\mathbb{R}\underline{h}$ *are fully faithful functors* $\mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^\wedge)$*.*

Sketch of proof. (1) follows from standard properties of mapping spaces (see [Hi]). For (2), we have

$$[x, y] \simeq \pi_0 Map_M(x, y) \simeq \pi_0 Hom_M(\Gamma(x)^*, Ry).$$

But

$$Hom_M(\Gamma(x)^*, Ry) \simeq \underline{h}_{Ry}(x) \simeq \underline{Hom}_{M^\wedge}(h_x, \underline{h}_{Ry}),$$

by the $SSet$-enriched Yoneda lemma; since h_x (resp., $\underline{h}_{Ry}$) is cofibrant (resp. fibrant) in $M^\wedge$, we have

$$\pi_0 \underline{Hom}_{M^\wedge}(h_x, \underline{h}_{Ry}) \simeq Hom_{\mathrm{Ho}(M^\wedge)}(h_x, \underline{h}_{Ry}),$$

because $M^\wedge$ is a simplicial model category. By (1), $h_x \simeq \underline{h}_{Rx}$ is an equivalence in $M^\wedge$, therefore

$$Hom_{\mathrm{Ho}(M^\wedge)}(h_x, \underline{h}_{Ry}) \simeq Hom_{\mathrm{Ho}(M^\wedge)}(\underline{h}_{Rx}, \underline{h}_{Ry}) \simeq Hom_{\mathrm{Ho}(M^\wedge)}(\mathbb{R}\underline{h}_x, \mathbb{R}\underline{h}_y).$$

□

Corollary 4.2.2 *For any* $F \in \mathrm{SPr}(M)$*,* $x \in M$ *we have an isomorphism in* $\mathrm{Ho}(SSet)$

$$\mathbb{R}\underline{Hom}(\underline{h}_x, F) \simeq F(x)$$

(where the right derivation in the l.h.s. is done with respect to the model structure $M^\wedge$*).*

Remark 4.2.3 Since

$$\begin{array}{ccc} \mathrm{Ho}(M) & \xrightarrow{\mathbb{R}\underline{h}} & \mathrm{Ho}(SPr(M)) \\ & \searrow^{\mathbb{R}\underline{h}} & \uparrow{\scriptstyle \mathbb{R}\mathrm{Id}} \\ & & \mathrm{Ho}(M^{\wedge}) \end{array}$$

is commutative up to a natural isomorphism (the natural transformation $\underline{h}_{R(-)} \longrightarrow R^{\wedge}(\underline{h}_{R(-)})$ is an isomorphism of functors $\mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(SPr(M))$, because $\underline{h}_{R(x)}$ is fibrant in $M^{\wedge}$ for any $x \in M$), we also get that $\mathbb{R}\underline{h} : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(SPr(M))$ is fully faithful.

Definition 4.2.4 *The functor(s):*

$$\mathbb{R}\underline{h} \simeq \mathrm{Ho}(h_{-}) : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^{\wedge})$$

will be called the model Yoneda embedding *for M.*

4.3 Model topologies

4.3.1 Model (pre-)topologies. Examples

To pass from *prestacks* to *stacks* over a model category, we need an appropriate notion of topology on a model category.
First, let us recall the definition of a Grothendieck (pre-)topology τ on a category $\mathcal{C}$:

- datum $\mathrm{Cov}_{\tau}(x)$ of a set of families of morphisms to x (called *τ-covering families*), for any $x \in \mathcal{C}$, subject to the following axioms:
  1. if $y \to x$ is an isomorphism, then the one element family $\{y \to x\}$ belongs to $\mathrm{Cov}_{\tau}(x)$;
  2. if $\{x_i \to x\} \in \mathrm{Cov}_{\tau}(x)$ and $y \longrightarrow x$ is a morphism in $\mathcal{C}$, then $\{x_i \times_x y \longrightarrow y\}$ exists and belongs to $\mathrm{Cov}_{\tau}(y)$;
  3. if $\{x_i \to x\} \in \mathrm{Cov}_{\tau}(x)$ and, for any i, we have $\{x_{ij} \to x_i\}_j \in \mathrm{Cov}_{\tau}(x_i)$, the composite family $\{x_i \to x\}_{i,j}$ belongs to $\mathrm{Cov}_{\tau}(x)$.

Now, if M is a model category, it is easy to modify conditions (1) to (3) above, in order to have properties that are invariant under weak equivalences: "isomorphisms" should be replaced with "isomorophisms in the homotopy category" and "fibered products" with "homotopy fibered products".

Definition 4.3.1 (Model pre-topology) *A* model pre-topology *τ on a model category M, consists of data $\{\mathrm{Cov}_{\tau}(x)\}_{x \in M}$ where $\mathrm{Cov}_{\tau}(x)$ consists of τ-covering families $\{x_i \longrightarrow x\}_i$ of maps in $\mathrm{Ho}(M)$ such that:*

1. *if $y \to x$ is an isomorphism in $\mathrm{Ho}(M)$, then the one element family $\{y \to x\}$ belongs to $\mathrm{Cov}_{\tau}(x)$;*
2. *if $\{x_i \to x\} \in \mathrm{Cov}_{\tau}(x)$ and $y \longrightarrow x$ is a morphism in $\mathcal{C}$, then $\left\{x_i \times^h_x y \longrightarrow y\right\}$ belongs to $\mathrm{Cov}_{\tau}(y)$;*
3. *if $\{x_i \to x\} \in \mathrm{Cov}_{\tau}(x)$ and, for any i, we have $\{x_{ij} \to x_i\}_j \in \mathrm{Cov}_{\tau}(x_i)$, the composite family $\{x_i \to x\}_{i,j}$ belongs to $\mathrm{Cov}_{\tau}(x)$.*

Note that (1) and (2) together imply that if (3) holds for a choice of a lift of

$$\begin{array}{ccc} & & x_i \\ & & \downarrow \\ y & \longrightarrow & x \end{array}$$

to M then it holds for any such choice.

Definition 4.3.2 *A pair (M, τ) where M is a model category and τ is a model topology on M, will be called a* model site.

Example 4.3.3

- $(M = Top, \pi_0 - surj)$.
  Take $\{X_i \longrightarrow X\} \in \mathrm{Cov}_\tau(x)$ iff $\coprod_i \pi_0 X_i \longrightarrow \pi_0 X$ is surjective.
- **Strong topologies on CDGA's.** Let k be a field of characteristic zero and $CDGA^{\leq 0}_{/k}$ be the category of non-positively graded commutative differential algebras. We will define the model site $(M := D - Aff_k) = ((CDGA^{\leq 0}_{/k})^{op}, \tau_0\text{-strong})$.
  Let τ_0 be any of the usual topologies defined over the category of k-schemes: Zariski, Nisnevich,étale, ffpf, ffqc,... Define the τ_0*-strong* model topology on M by $\{B \longrightarrow A_i\} \in \mathrm{Cov}_{\tau_0\text{-strong}}(B)$ iff
  - $\left\{\mathrm{Spec} H^0(A_i) \longrightarrow \mathrm{Spec} H^0(B)\right\}$ is a τ_0-covering in $(Sch_{/k})$;
  - for any $i \in I$, $H^*(A_i) \simeq H^*(B) \otimes_{H^0(A_i)} H^0(B)$.

  i.e., morally speaking, τ_0-strong is the τ_0-topology on the zero-truncation or scheme-like "direction" and everything else is just a pull-back.
  In the case $\tau_0 :=$ (ét), this topology was first introduced by Kai Behrend. This topology will be used in the next lectures to describe applications of homotopical algebraic geometry to *derived moduli spaces.*
  If one wishes to work over an arbitrary ground field or ring R, one has to replace $CDGA^{\leq 0}_{/k}$ with the category $E_\infty - Alg_R$ of E_∞-algberas over $Ch(R)$: the same topologies make sense. Note also that if $R = k$ and k is a field of caharacteristic zero, then $\mathrm{Ho}(E_\infty - Alg_R) \simeq \mathrm{Ho}(CDGA^{\leq 0}_{/k})$.
- **Semistrong topologies on CDGA's**. Same set-up as in the previous example but the two conditions reduce to just
  - $\left\{\mathrm{Spec} H^0(A_i) \longrightarrow \mathrm{Spec} H^0(B)\right\}$ is a τ_0-covering in $(Sch_{/k})$.
- $Tor_{\geq 0}$**-topology on unbounded CDGA's**. Let $M = (CDGA^{unb}_{/k})^{op}$ be the (opposite) category of unbounded cdga's over a field k of characteristic zero; we will define the *positive Tor-dimension* model topology $Tor_{\geq 0}$ on M. Let's stipulate $\{f_i : B \longrightarrow A_i\} \in \mathrm{Cov}_{Tor_{\geq 0}}(B)$ iff
  - for any $i \in I$, the homotopy base change functor
  $$\mathbb{L}f_i^* = (-) \otimes^{\mathbb{L}}_B A_i : \mathrm{Ho}(Mod_B) \longrightarrow \mathrm{Ho}(Mod_{A_i})$$
    preserves the subcategory of *cohomologically positive modules* (i.e., modules P, such that $H^i(P) = 0$, for any $i \leq 0$);
  - (Covering condition) the family of derive base change functors
  $$\left\{\mathbb{L}f_i^* = (-) \otimes^{\mathbb{L}}_B A_i : \mathrm{Ho}(Mod_B) \longrightarrow \mathrm{Ho}(Mod_{A_i})\right\}$$
    is *conservative.*

  This topology is relevant for putting B. Toen's *higher tannakian duality* ([To]) in the framework of *algebriac geometry over the category of complexes.*

4.3.2 Relation between model topologies on M and Grothendieck topologies on Ho(M)

Let τ be a model topology on a model category M. Define a sieve $\mathcal{R}$ over x in $\mathrm{Ho}(M)$ a τ*-covering sieve* if it contains a τ-covering family. One easily gets:

Proposition 4.3.4 *The τ-covering sieves form a Grothendieck topology, denoted by $\underline{\tau}$, on* $\mathrm{Ho}(M)$.

Conversely, given a Grothendieck topology σ on $\mathrm{Ho}(M)$, define $\{x_i \longrightarrow x\} \in \mathrm{Cov}_{\overline{\sigma}}(x)$ iff the sieve it generates is a σ-covering sieve.

Proposition 4.3.5 *The data $\{\mathrm{Cov}_{\overline{\sigma}}(x)\}$ define a model topology on M, denoted by $\overline{\sigma}$.*

The two constructions, as usual for pretopologies/topologies, are *almost* inverse to each other:

Proposition 4.3.6 *Let us call a model topology on M* saturated *if any family of maps containing a covering family, is again a covering family.*
Then, the maps $\tau \longmapsto \underline{\tau}$ and $\sigma \longmapsto \overline{\sigma}$ define a bijection

$$\{\text{saturated model pretopologies on } M\} \simeq \{\text{Grothendieck topologies on } \mathrm{Ho}(M)\}.$$

4.4 Model category of stacks on a model site

Let (M, τ) be a model site and $M^{\wedge} := L_{h_W}(\mathrm{SPr}(M))$ be the model category of pre-stacks on M.
The model structure $M^{\wedge}$ does not "see"the model topology τ; so, like we did in the non-model case (Lecture 4) in passing from $\mathrm{SPr}(\mathcal{C})$ to $\mathrm{SPr}(\mathcal{C})_{\mathrm{loc}}$, we homotopically "invert" analogs of τ-hypercovers (called *homotopy τ-hypercovers*) by means of a further left Bousfield localization[3]. So we pass from $M^{\wedge}$ to $M^{\sim,\tau}$, which is called the *model category of stacks on the model site* (M, τ).
Pretty much like in the non-model case (Lecture 4), we have natural notions of *sheaves of homotopy groups* (defined as sheaves on the induced Grothendieck site $(\mathrm{Ho}(M), \underline{\tau})$).

Definition 4.4.1 *A map $F \longrightarrow G$ of simplicial presheaves on M is called a* π_*-equivalence *if it induces isomorphisms on all the sheaves of higher homotopy groups (for any choice of "'base point").*

Theorem 4.4.2 (Recognition principle for weak equivalences in $M^{\sim,\tau}$)
Weak equivalences in $M^{\sim,\tau}$ are exactly π_*-equivalences.

The standard properties of the left Bousfield localization (see Lecture 3) imply that $\mathrm{Ho}(M^{\sim,\tau})$ can be canonically identified with the full subcategory of $\mathrm{Ho}(\mathrm{SPr}(M))$ of objects F:

- sending weak equivalences in M to weak equivalences in $SSet$;
- satisfying the *τ-hyperdescent condition*

$$F(X) \widetilde{\longrightarrow} holim F(U_\bullet),$$

  for any homotopy τ-hypercover $U_\bullet \longrightarrow X$ (in the "small" set mentioned above).

Definition 4.4.3 *Objects in* $\mathrm{Ho}(M^{\sim,\tau})$ *will be called* stacks on the model site (M, τ).

Hom-stacks. Like presheaves or sheaves on a topologicl space have *internal Hom's*, the same is true for stacks on a model site (compare with Lecture 4): $\mathrm{Ho}(M^{\sim,\tau})$ is a *cartesian closed category* i.e., for any $F, G \in \mathrm{Ho}(M^{\sim,\tau})$ there exists a stack $\mathcal{H}om(F, G) \in \mathrm{Ho}(M^{\sim,\tau})$, satisfying the usual adjunction properties.

Functorialities. For functorialities (direct/inverse images) of the category of stacks with respect to morphisms of model categorie/sites, we refer to [HAG-I, §4.8].

[3] technically speaking, we need to find a *subset* of all the homotopy hypercovers that has a reasonably small size.

4.5 Analogy with topos theory

Let us reinterprete usaul *sheaf theory* in a way that will make it transparent the analogy with the construction of the category of stacks over a model site.

- $(\mathcal{C}, \tau)$: Grothendieck site.
- $\mathrm{Pr}(\mathcal{C})$: category of presheaves (of sets) on $\mathcal{C}$.
- Yoneda embedding: $h : \mathcal{C} \longrightarrow \mathrm{Pr}(\mathcal{C})$
- $\Sigma_\tau :=$ τ-local isomorphisms in $\mathrm{Pr}(\mathcal{C})$ (by definition, maps which are surjective and injective up to a τ-covering-refinement).
- Localization $a := \mathrm{loc} : \mathrm{Pr}(\mathcal{C}) \longrightarrow (\Sigma_\tau)^{-1}\mathrm{Pr}(\mathcal{C}) := \mathrm{Sh}(\mathcal{C}, \tau)$.

The basic properties are collected in the following:

Theorem A.

1. The category $\mathrm{Sh}(\mathcal{C}, \tau)$ has *all limits and colimits*;
2. The localization functor $a := \mathrm{loc} : \mathrm{Pr}(\mathcal{C}) \longrightarrow \mathrm{Sh}(\mathcal{C}, \tau)$ is *left exact* (i.e. commutes with finite limits) and *has a fully-faithful right adjoint* $j : \mathrm{Sh}(\mathcal{C}, \tau) \longrightarrow \mathrm{Pr}(\mathcal{C})$.
3. The category $\mathrm{Sh}(\mathcal{C}, \tau)$ is *cartesian closed*, i.e. it has internal Hom's (Hom-sheaves).

Of course the essential image of j consists of usual sheaves, i.e. presheaves having the τ-descent property, and the localization functor a becomes equivalent to the usual *associated-sheaf* functor. In a way, the definition above was like "defining the category of sheaves without defining what a sheaf is".

Let us now turn to the *stacks over model sites* part of the picture.

- (M, τ): model site.
- $M^\wedge$: category of pre-stacks on M.
- Model Yoneda embedding: $\mathbb{R}\underline{h} \simeq \mathrm{Ho}(h_-) : \mathrm{Ho}(M) \longrightarrow \mathrm{Ho}(M^\wedge)$.
- $S_\tau :=$ π_*-equivalences (or hhc_τ-local equivalences).
- Homotopy localization $\mathrm{Id} : M^\wedge \longrightarrow L_{S_\tau}(M^\wedge) := M^{\sim,\tau}$.

The basic properties are collected in

Theorem A-model.

1. The category $M^{\sim,\tau}$ is a model category, therefore (Lecture 3) it has *all homotopy limits and homotopy colimits*;
2. The left derived functor $\mathbb{L}\mathrm{Id} : \mathrm{Ho}(M^\wedge) \longrightarrow \mathrm{Ho}(M^{\sim,\tau})$ is *homotopy left exact* (i.e. commutes with homotopy fibered products) and *has a fully-faithful right adjoint* $\mathbb{R}\mathrm{Id} : \mathrm{Ho}(M^{\sim,\tau}) \longrightarrow \mathrm{Ho}(M^\wedge)$.
3. The category $\mathrm{Ho}(M^{\sim,\tau})$ is *cartesian closed*, i.e. it has internal Hom-stacks.

5 Lectures 7 and 8. Derived Moduli Spaces.

The final lectures 7 and 8 are devoted to an application of HAG (Homotopical Algebraic Geometry, as explained in Lectures 5 and 6) to DAG (Derived Algebraic geometry), especially to the description of *derived moduli spaces*.
The following text is [HAGDAG], available as preprint math.AG/0210407 in the arXiv.

5.1 What's HAG ?

Homotopical Algebraic Geometry *(or* ***HAlgebraic Geometry****, or simply* ***HAG****) was conceived as a framework to talk about schemes in a context where affine objects are in one-to-one correspondence with commutative monoid-like objects in a base symmetric monoidal model category.*

This general definition might seem somewhat obscure, so we'd rather mention the most important examples of base symmetric monoidal model category, and the corresponding notion of commutative monoid-like objects. In each of the following situations, HAG will provide a context in which one can *do algebraic geometry* (and in particular, talk about schemes, algebraic spaces, stacks ...), hence giving rise to various *geometries.*

1. The model category Ab of abelian groups (with its trivial model structure) and the tensor product of abelian groups. Commutative monoid objects are commutative rings. The corresponding geometry is the usual, Grothendieck-style algebraic geometry.
2. The model category $Mod(\mathcal{O})$ of $\mathcal{O}$-modules over some ringed site (with the trivial model structure) and the tensor product of $\mathcal{O}$-modules. Commutative monoid objects are sheaves of commutative $\mathcal{O}$-algebras. The corresponding geometry is called *relative algebraic geometry*, and was introduced and studied in [Ha, De].
3. The model category $C(k)$ of complexes over some ring k and the tensor product of complexes (see [Ho, §2.3]). Commutative monoid-like objects are commutative E_∞-algebras over k ([Kr-Ma]). The corresponding geometry is the so-called *derived algebraic geometry* that we are going to discuss in details in this paper, and for which one possible avatar is the theory of dg-schemes and dg-stacks of [Ci-Ka1, Ci-Ka2].
4. The model category Sp of symmetric spectra and the smash product (see [Ho-Sh-Sm]), or equivalently the category of $\mathbb{S}$-modules (see [EKMM]). Commutative monoid-like objects are E_∞-ring spectra, or commutative $\mathbb{S}$-algebras. We call the corresponding geometry *brave new algebraic geometry*, quoting the expression *brave new algebra* introduced by F. Waldhausen (for more details on the subject, see e.g. [Vo]).
5. The model category Cat of categories and the direct product (see, e.g. [Jo-Ti]). Commutative monoid-like objects are symmetric monoidal categories. The corresponding geometry does not have yet a precise name, but could be called *2-algebraic geometry*, since vector bundles in this setting will include both the notion of 2-vector spaces (see [Ka-Vo]) and its generalization to 2-*vector bundles.*

For the general framework, we refer the reader to [HAG-I, HAG-II]. The purpose of the present note is to present one possible incarnation of HAG through a concrete application to *derived algebraic geometry* (or "DAG" for short).

5.2 What's DAG ?

Of course, the answer we give below is our own limited understanding of the subject.

As far as we know, the foundational ideas of *derived algebraic geometry* (whose infinitesimal theory is also referred to as *derived deformation theory*, or "DTT" for short) were introduced by P. Deligne, V. Drinfel'd and M. Kontsevich, for the purpose of studying the so-called *derived moduli spaces.* One of the main observation was that certain moduli spaces were very *singular* and not of the *expected dimension*, and according to the general philosophy this was considered as somehow unnatural (see the *hidden smoothness philosophy* presented in [Ko1]). It was therefore expected that these moduli spaces are only *truncations* of some richer geometric objects, called the *derived moduli spaces*, containing important additional structures making them *smooth and of the expected dimension.* In order to illustrate these general ideas, we present here the fundamental example of the moduli stack of vector bundles (see the introductions of [Ci-Ka1, Ci-Ka2, Ka1] for more motivating examples as well as philosophical remarks).

Let C be a smooth projective curve (say over $\mathbb{C}$), and let us consider the moduli stack $\underline{Vect}_n(C)$ of rank n vector bundles on C (here $\underline{Vect}_n(C)$ classifies all vector bundles on C, not only the semi-stable or stable ones). The stack $\underline{Vect}_n(C)$ is known to be an algebraic stack (in the sense of Artin). Furthermore, if $E \in \underline{Vect}_n(C)(\mathbb{C})$ is a vector bundle on C, one can easily compute the *stacky tangent space* of $\underline{Vect}_n(C)$ at the point E. This *stacky tangent space* is actually a complex of $\mathbb{C}$-vector spaces concentrated in degrees $[-1,0]$, which is easily seen to be quasi-isomorphic to the complex $C^*(C_{Zar}, \underline{End}(E))[1]$ of Zariski cohomology of C with coefficient in the vector bundle $\underline{End}(E) = E \otimes E^*$. Symbolically, one writes

$$T_E\underline{Vect}(C) \simeq H^1(C, \underline{End}(E)) - H^0(C, \underline{End}(E)).$$

This implies in particular that the *dimension* of $T_E\underline{Vect}(C)$ is independent of the point E, and is equal to $n^2(g-1)$, where g is the genus of C. The conlcusion is then that the stack $\underline{Vect}_n(C)$ is smooth of dimension $n^2(g-1)$.

Let now S be a smooth projective surface, and $\underline{Vect}_n(S)$ the moduli stack of vector bundles on S. Once again, $\underline{Vect}_n(S)$ is an algebraic stack, and the stacky tangent space at a point $E \in \underline{Vect}_n(S)(\mathbb{C})$ is easily seen to be given by the same formula

$$T_E\underline{Vect}_n(S) \simeq H^1(S, \underline{End}(E)) - H^0(S, \underline{End}(E)).$$

Now, as $H^2(S, \underline{End}(E))$ might jump when specializing E, the dimension of $T_E\underline{Vect}(S)$, which $h^1(S, \underline{End}(E)) - h^0(S, \underline{End}(E))$, is not locally constant and therefore the stack $\underline{Vect}_n(S)$ is not smooth anymore.

The main idea of *derived algebraic geometry* is that $\underline{Vect}_n(S)$ is only the truncation of a richer object $\mathbb{R}\underline{Vect}_n(S)$, called the *derived moduli stack of vector bundles on S*. This derived moduli stack, whatever it may be, should be such that its *tangent space* at a point E is the *whole* complex $C^*(S, \underline{End}(E))[1]$, or in other words,

$$T_E\mathbb{R}\underline{Vect}_n(S) \simeq -H^2(S, \underline{End}(E)) + H^1(S, \underline{End}(E)) - H^0(S, \underline{End}(E)).$$

The dimension of its tangent space at E is then expected to be $-\chi(S, \underline{End}(E))$, and therefore locally constant. Hence, the object $\mathbb{R}\underline{Vect}_n(S)$ is expected to be *smooth*.

Remark 5.2.1 Another, very similar but probably more striking example is given by the moduli stack of stable maps, introduced in [Ko1]. A consequence of the expected existence of the *derived moduli stack of stable maps* is the presence of a *virtual structure sheaf* giving rise to a *virtual fundamental class* (see [Be-Fa]). The importance of such constructions in the context of Gromov-Witten theory shows that the extra information contained in *derived moduli spaces* is very interesting and definitely geometrically meaningful.

In the above example of the stack of vector bundles, the tangent space of $\mathbb{R}Vect_n(S)$ is expected to be a complex concentrated in degree $[-1,1]$. More generally, one can get convinced that tangent spaces of derived moduli (1-)stacks should be complexes concentrated in degree $[-1,\infty[$ (see [Ci-Ka1]). It is therefore pretty clear that in order to make sense of an object such as $\mathbb{R}\underline{Vect}_n(S)$, schemes and algebraic stacks are not enough, and one should look for a more general definition of *spaces*. This leads to the following general question.

Problem: *Provide a framework in which* derived moduli stacks *can actually be constructed. In particular, construct the derived moduli stack of vector bundles* $\mathbb{R}\underline{Vect}(S)$ *discussed above.*

Several construction of *formal* derived moduli spaces have appeared in the litterature (see for example [Ko-So, So]), a general framework for *formal DAG* have been developed by V. Hinich in [Hin2], and pro-representability questions were investigated by Manetti in [Man]. So, in a sense, the formal theory has already been worked out, and what remains of the problem above is an approach to *global DAG*.

A first approach to the global theory was proposed by M. Kapranov and I. Ciocan-Fontanine, and is based on the theory of *dg-schemes* or more generally of *dg-stacks* (see [Ci-Ka1, Ci-Ka2]). A dg-scheme is, roughly speaking, a scheme together with an enrichement of its structural sheaf into commutative differential graded algebras. This enriched structural sheaf is precisely the datum encoding the *derived* information.

This approach has been very successful, and many interesting derived moduli spaces (or stacks) have already been constructed as dg-schemes (e.g. the derived version of the Hilbert scheme, of the Quot scheme, of the stack of stable maps, and of the stack of local systems on a space have been defined in [Ka2, Ci-Ka1, Ci-Ka2]). However, this approach have encountered two major problems, already identified in [Ci-Ka2, 0.3].

1. The definition of dg-schemes and dg-stacks seems too rigid for certain purposes. By definition, a dg-scheme is a space obtained by *gluing commutative differential graded algebras for the Zariski topology*. It seems however that certain constructions really require a weaker notion of gluing, as for example *gluing differential graded algebras up to quasi-isomorphisms*.
2. The notion of dg-schemes is not very well suited with respect to the functorial point of view, as representable functors would have to be defined on the derived category of dg-schemes (i.e. the category obtained by formally inverting quasi-ismorphisms of dg-schemes), which seems difficult to describe and to work with. As a consequence, the derived moduli spaces constructed in [Ka2, Ci-Ka1, Ci-Ka2] do not arise as solution to natural *derived moduli problems*, and are constructed in a rather ad-hoc way.

The first of these difficulties seems of a technical nature, whereas the second one seems more fundamental. It seems a direct consequence of these two problems that the derived stack of vector bundles still remains to be constructed in this framework (see [Ka1] and [Ci-Ka1, Rem. 4.3.8]).

It is the purpose of this note to show how HAG might be applied to provide a framework for DAG in which problems (1) and (2) hopefully disappear. We will show in particular how to make sense of various *derived moduli functors* whose *representability* can be proved in many cases.

5.3 The model category of D-stacks

In this subection we will present the construction of a *model category of D-stacks*. It will be our derived version of the category of stacks that is commonly used in moduli theory, and all our examples of derived moduli stacks will be objects of this category.

The main idea of the construction is the one used in [HAG-I], and consists of adopting systematically the functorial point of view. Schemes, or stacks, are sheaves over the category of commutative algebras. In the same way, D-stacks will be *sheaves-like objects* on the category of commutative differential graded algebras. This point of view may probably be justified if one convinces himself that commutative differential graded algebras *have to be* the affine derived moduli spaces, and that therefore they are the elementary pieces of the theory that one would like to glue to obtain global geometric objects. Another, more down to earth, justification would just be to notice that all of the *wanted derived moduli spaces* we are aware of, have a reasonable model as an object in our category of D-stacks.

Before starting with the details of the construction, we would like to mention that K. Behrend has independently used a similar approach to DAG that uses the 2-category of differential graded algebras (see [Be]) (his approach is actually the 2-truncated version of ours). It is not clear to us that the constructions and results we are going to present in this work have reasonable analogs in his framework, as they use in an essential way higher homotopical informations that are partially lost when using any truncated version.

Conventions. For the sake of simplicity, we will work over the field of complex numbers $\mathbb{C}$. The expression *cdga* will always refer to a *non-positively graded commutative differential graded algebra* over $\mathbb{C}$ with differential of degree 1. Therefore, a cgda A looks like

$$\cdots \longrightarrow A^{-n} \longrightarrow A^{-n+1} \longrightarrow \cdots \longrightarrow A^{-1} \longrightarrow A^{0}.$$

The category CDGA of cgda's is endowed with its usual model category structure (see e.g. [Hin1]), for which fibrations (resp. equivalences) are epimorphisms in degree ≤ -1 (resp. quasi-isomorphisms).

5.3.1 D-Pre-stacks

With start by defining $D - Aff := CDGA^{op}$ to be the opposite category of cdga's, and we consider the category $SPr(D - Aff)$, of simplicial presheaves on $D - Aff$, or equivalently the category of functors from $CDGA$ to $SSet$. The category $SPr(D - Aff)$ is endowed with its objectwise projective model structure in which fibrations and equivalences are defined objectwise (see [Hi, 13.10.17]).

For any cdga $A \in D - Aff$, we have the presheaf of sets represented by A, denoted by

$$\begin{array}{rccc} h_A : & D - Aff^{op} & \longrightarrow & Set \\ & B & \mapsto & Hom(B, A). \end{array}$$

Considering a set as a constant simplicial set, we will look at h_A as an object in $SPr(D - Aff)$. The construction $A \mapsto h_A$ is clearly functorial in A, and therefore for any $u : A \to A'$ in $D - Aff$, corresponding to a quasi-isomorphism of cdga's, we get a morphism $u : h_A \to h_{A'}$ in $SPr(D - Aff)$. These morphisms will simply be called *quasi-isomorphisms.*

Definition 5.3.1 *The* model category of D-pre-stacks *is the left Bousfield localization of the model category $SPr(D - Aff)$ with respect to the set of morphisms $\{u : h_A \to h_{A'}\}$, where u varies in the set of all quasi-isomorphisms. It will be denoted by $D - Aff^{\wedge}$.*

Remark 5.3.2 *1.* The careful reader might object that the category $D - Aff$ and the set of all quasi-isomorphisms are not small, and therefore that definition 5.3.1 does not make sense. If this happens (and only then), take two universes $\mathbb{U} \in \mathbb{V}$, define $CDGA$ as the category of $\mathbb{U}$-small cdga's and $SPr(D - Aff)$ as the category of functors from $CDGA$ to the category of $\mathbb{V}$-small simplicial sets. Definition 5.3.1 will now make sense.

2. In [HAG-I], the model category $D - Aff^{\wedge}$ was denoted by $(D - Aff, W)^{\wedge}$, where W is the subcategory of quasi-isomorphisms.

By general properties of left Bousfield localization (see [Hi]), the fibrant objects in $D-Aff^{\wedge}$ are the functors $F : CDGA \longrightarrow SSet$ satisfying the following two conditions

1. For any $A \in CDGA$, the simplicial set $F(A)$ is fibrant.
2. For any quasi-isomorphism $u : A \longrightarrow B$ in CDGA, the induced morphism $F(u) : F(A) \longrightarrow F(B)$ is a weak equivalence of simplicial sets.

¿From this description, we conclude in particular, that the homotopy category $\mathrm{Ho}(D-Aff^{\wedge})$ is naturally equivalent to the full sub-category of $\mathrm{Ho}(SPr(D-Aff))$ consisting of functors $F : CDGA \longrightarrow SSet$ sending quasi-isomorphisms to weak equivalences. We will use implicitely this description, and we will always consider $\mathrm{Ho}(D-Aff^{\wedge})$ as embedded in $\mathrm{Ho}(SPr(D-Aff))$.

Definition 5.3.3 *Objects of $D-Aff^{\wedge}$ satisfying condition* (2) *above (i.e. sending quasi-isomorphisms to weak equivalences) will be called* D-pre-stacks.

5.3.2 D-Stacks

Now that we have constructed the model category of D-pre-stacks we will introduce some kind of *étale topology* on the category $D-Aff$. This will allow us to talk about a corresponding notion of étale *local equivalences* in $D-Aff^{\wedge}$, and to define the model category of D-stacks by including the *local-to-global principle* into the model structure.

We learned the following notion of formally étale morphism of cdga's from K. Behrend.

Definition 5.3.4 *A morphism $A \longrightarrow B$ in $CDGA$ is called* formally étale *if it satisfies the following two conditions.*

1. *The induced morphism $H^0(A) \longrightarrow H^0(B)$ is a formally étale morphism of commutative algebras.*
2. *For any $n < 0$, the natural morphism of $H^0(B)$-modules*

$$H^n(A) \otimes_{H^0(A)} H^0(B) \longrightarrow H^n(B)$$

is an isomorphism.

Remark 5.3.5 Its seems that a morphism $A \longrightarrow B$ of cdga's is formally étale in the sense of Definition 5.3.4 if and only if the relative cotangent complex $\mathbb{L}\Omega^1_{B/A}$ (e.g. in the sense of [Hin1]) is acyclic. This justifies the terminology.

¿From Definition 5.3.4 we now define the notion of étale covering families. For this, we recall that a morphism of cdga's $A \longrightarrow B$ is said to be *finitely presented* if B is equivalent to a retract of a finite cell A-algebra (see for example [EKMM]). This is also equivalent to say that for any filtered systems $\{A \longrightarrow C_i\}_{i\in I}$, the natural morphism

$$Colim_{i\in I}\mathrm{Map}_{A/CDGA}(B, C_i) \longrightarrow \mathrm{Map}_{A/CDGA}(B, Colim_{i\in I}C_i)$$

is a weak equivalence (here $\mathrm{Map}_{A/CDGA}$ denotes the mapping spaces, or function complexes, of the model category $A/CDGA$ of cdga's under A, as defined in [Ho, §5.4])[4].

[4] We warn the reader that if commutative algebras are considered as cdga's concentrated in degree zero, the notion of finitely presented morphisms of commutative algebras and the notion of finitely presented morphisms of cdga's are *not the same.* In fact, for a morphism of commutative algebras it is stronger to be finitely presented as a morphism of cdga's than as a morphism of algebras.

Definition 5.3.6 *A finite family of morphisms of cdga's*

$$\{A \longrightarrow B_i\}_{i\in I}$$

is called an étale covering *if it satisfies the following three conditions*

1. *For any $i \in I$, the morphism $A \longrightarrow B_i$ is finitely presented.*
2. *For any $i \in I$, the morphism $A \longrightarrow B_i$ is formally étale.*
3. *The induced family of morphisms of affine schemes*

$$\{Spec\, H^0(B_i) \longrightarrow Spec\, H^0(A)\}_{i\in I}$$

   is an étale covering.

The above definition almost defines a pre-topology on the category $D-Aff$. Indeed, stability and composition axioms for a pre-topology are statisfied, but the base change axiom is not. In general, the base change of an étale covering $\{A \longrightarrow B_i\}_{i\in I}$ along a morphism of $A \longrightarrow C$ will only be an étale covering if $A \longrightarrow C$ is a cofibration in CDGA. In other words, for the base change axiom to be satisfied one needs to replace fibered products by homotopy fibered products in $D-Aff$. Therefore, the étale covering families of Definition 5.3.6 do not satisfy the axioms for a pre-topology on $D-Aff$, but rather satisfy a *homotopy analog* of them. This is an example of a *model pre-topology* on the model category $D-Aff$, for which we refer the reader to [HAG-I, §4.3] where a precise definition is given.

In turns out that the data of a model pre-topology on a model category M is more or less equivalent to the data of a Grothendieck topology on its homotopy category $\mathrm{Ho}(M)$ (see [HAG-I, Prop. 4.3.5]). In our situation, the étale coverings of Definition 5.3.6 induce a Grothendieck topology, called the *étale topology* on the opposite of the homotopy category $\mathrm{Ho}(D-Aff)$ of cdga's. More concretely, a sieve S over a cdga $A \in \mathrm{Ho}(D-Aff)$ is declared to be a covering sieve if it contains an étale covering family $\{A \longrightarrow B_i\}_{i\in I}$. The reader will check as an exercise that this defines a topology on $\mathrm{Ho}(D-Aff)$ (hint: one has to use that étale covering families are stable by homotopy pull-backs in $D-Aff$, or equivalentely by homotopy push-outs in $CDGA$). From now on, we will always consider $\mathrm{Ho}(D-Aff)$ as a Grothendieck site for this étale topology.

For a D-pre-stack $F \in D-Aff^{\wedge}$ (recall from Definition 5.3.3 that this implies that F sends quasi-isomorphisms to weak equivalences), we define its presheaf of connected components

$$\begin{array}{cccc} \pi_0^{pr}(F): & D-Aff^{op} & \longrightarrow & Set \\ & A & \mapsto & \pi_0(F(A)). \end{array}$$

As the object F is a D-pre-stack (see 5.3.3), the functor $\pi_0^{pr}(F)$ will factors through the homotopy category

$$\begin{array}{cccc} \pi_0^{pr}(F): & \mathrm{Ho}(D-Aff)^{op} & \longrightarrow & Set \\ & A & \mapsto & \pi_0(F(A)). \end{array}$$

We can consider the sheaf $\pi_0(F)$ associated to the presheaf π_0^{pr} in the étale topology on $\mathrm{Ho}(D-Aff)$. The sheaf $\pi_0(F)$ is called the 0-*homotopy sheaf* of the D-pre-stack F. Now, if $F \in D-Aff^{\wedge}$ is any simplicial presheaf, then one can apply the above construction to one of its fibrant models RF. This allows us to define its 0-th homotopy sheaf as $\pi_0(F) := \pi_0(RF)$.

As for the case of simplicial presheaves (see [Ja1]), one can also define higher homotopy sheaves, which are sheaves of groups and abelian groups on the sites $\mathrm{Ho}(D-Aff/A)$ for various cdga's A.

Precisely, let F be a D-pre-stacks and $s \in F(A)_0$ a point over a cdga $A \in D-Aff$. We define the n-th homotopy group presheaf pointed at s by

$$\begin{array}{cccc} \pi_n^{pr}(F,s): & D-Aff^{op}/A = A/CDGA & \longrightarrow & Gp \\ & (u: A \to B) & \mapsto & \pi_n(F(B), u^*(s)). \end{array}$$

Again, as F is a D-pre-stack, this presheaves descend to the homotopy category

$$\begin{array}{cccc} \pi_n^{pr}(F,s): & \mathrm{Ho}(D-Aff^{op}/A) = \mathrm{Ho}(A/CDGA) & \longrightarrow & Gp \\ & (u: A \to B) & \mapsto & \pi_n(F(B), u^*(s)). \end{array}$$

The étale model pre-topology on $D-Aff$ also induces Grothendieck topologies on the various homotopy categories $\mathrm{Ho}(A/CDGA)$, and therefore one can consider the sheaves associated to $\pi_n^{pr}(F,s)$. These sheaves are called the *n-th homotopy sheaves* of F pointed at s and are denoted by $\pi_n(F,s)$. As before, if F is any object in $D-Aff^\wedge$, one can define $\pi_n(F,s) := \pi_n(RF,s)$ for RF a fibrant replacement of F.

The notion of homotopy sheaves defined above gives rise to the following notion of local equivalences.

Definition 5.3.7 *A morphism $f: F \longrightarrow F'$ in $D-Aff^\wedge$ is called a* local equivalence *if it satisfies the following two conditions*

1. *The induced morphism of sheaves $\pi_0(F) \longrightarrow \pi_0(F')$ is an isomorphism.*
2. *For any $A \in D-Aff$, and any point $s \in F(A)$, the induced morphism of sheaves $\pi_n(F,s) \longrightarrow \pi_n(F', f(s))$ is an isomorphism.*

One of the key results of "HAG" is the following theorem. It is a very special case of the existence theorem [HAG-I, §4.6], which extends the existence of the local model structure on simplicial presheaves (see [Ja1]) to the case of model sites.

Theorem 5.3.8 *There exists a model category structure on $D-Aff^\wedge$ for which the equivalences are the local equivalences and the cofibrations are the cofibrations in the model category $D-Aff^\wedge$ of D-pre-stacks.*

This model category is called the model category of D-stacks *for the étale topology, and is denoted by $D-Aff^\sim$.*

The reason for calling $D-Aff^\sim$ the model category of D-stacks is the following proposition. It follows from [HAG-I, 4.6.3], which is a generalization to model sites of the main theorem of [DHI].

Proposition 5.3.9 *An object $F \in D-Aff^\sim$ is fibrant if and only if it satisfies the following three conditions*

1. *For any $A \in D-Aff$, the simplicial set $F(A)$ is fibrant.*
2. *For any quasi-isomorphism of cdga's $A \longrightarrow B$, the induced morphism $F(A) \longrightarrow F(B)$ is a weak equivalence.*
3. *For any cdga A, and any étale hyper-covering in $D-Aff$ (see [HAG-I] for details) $A \longrightarrow B_*$, the induced morphism*
$$F(A) \longrightarrow Holim_{n\in\Delta} F(B_n)$$
is a weak equivalence.

Condition (3) is called the *stack condition for the étale topology.* Note that a typical étale hyper-covering of cdga's $A \longrightarrow B_*$ is given by the homotopy co-nerve of an étale covering morphism $A \longrightarrow B$

$$B_n := \underbrace{B \otimes^{\mathbb{L}}_A B \otimes^{\mathbb{L}}_A \cdots \otimes^{\mathbb{L}}_A B}_{n \ times}.$$

Condition (3) for these kind of hyper-coverings is the most commonly used descent condition, but as first shown in [DHI] requiring descent with respect to *all* étale hyper-coverings is necessary for Proposition 5.3.9 to be correct.

Definition 5.3.10 *A* D-stack *is any object* $F \in D - Aff^{\sim}$ *satisfying conditions* (2) *and* (3) *of Proposition 5.3.9. By abuse of language, objects in the homotopy category* $\mathrm{Ho}(D - Aff^{\sim})$ *will also be called* D*-stacks.*

A morphism *of* D*-stacks is a morphism in the homotopy category* $\mathrm{Ho}(D - Aff^{\sim})$.

The second part of the definition is justified because the homotopy category $\mathrm{Ho}(D - Aff^{\sim})$ is naturally equivalent to the full sub-category of $\mathrm{Ho}(SPr(D - Aff))$ consisting of objects satisfying conditions (2) and (3) of Proposition 5.3.9.

5.3.3 Operations on D-stacks

One of the main consequences of the existence of the model structure on $D - Aff^{\sim}$ is the possibility to define several standard operations on D-stacks, analogous to the ones used in sheaf theory (limits, colimits, sheaves of morphisms ...).

First of all, the category $D - Aff^{\sim}$ being a category of simplicial presheaves, it comes with a natural enrichement over the category of simplicial sets. This makes $D - Aff^{\sim}$ into a simplicial model category (see [Ho, 4.2.18]). In particular, one can define in a standard way the *derived simplicial Hom's* (well defined in the homotopy category $\mathrm{Ho}(SSet)$),

$$\mathbb{R}\underline{Hom}(F,G) := \underline{Hom}(QF, RG),$$

where Q is a cofibrant replacement functor, R is a fibrant replacement functor, and $\underline{Hom}$ are the simplicial Hom's sets of $D - Aff^{\sim}$. These derived simplicial Hom's allows one to consider spaces of morphisms between D-stacks, in the same way as one commonly considers groupoids of morphisms between stacks in groupoids (see [La-Mo]).

This simplicial structure also allows one to define *exponentials* by simplicial sets. For an object $F \in D - Aff^{\sim}$ and $K \in SSet$, one has a well defined object in $\mathrm{Ho}(D - Aff^{\sim})$

$$F^{\mathbb{R}K} := (RF)^K$$

which satisfies the usual adjunction formula

$$\mathbb{R}\underline{Hom}(G, F^{\mathbb{R}K}) \simeq \mathbb{R}\underline{Hom}(K, \mathbb{R}\underline{Hom}(G,F)).$$

The existence of the model structure $D - Aff^{\sim}$ also implies the existence of *homotopy limits* and *homotopy colimits*, as defined in [Hi, §19]. The existence of these homotopy limits and colimits is the analog of the fact that category of sheaves have all kind of limits and colimits. We will use in particular homotopy pull-backs i.e. homotopy limits of diagrams $F \longleftarrow H \longrightarrow G$, that will be denoted by

$$F \times^h_H G := Holim\{\, F \longleftarrow H \longrightarrow G \,\}.$$

Finally, one can show that the homotopy category $\mathrm{Ho}(D-Aff^{\sim})$ is *cartesian closed* (see [HAG-I, §4.7]). Therefore, for any two object F and G, there exists an object $\mathbb{R}\mathcal{HOM}(F,G) \in \mathrm{Ho}(D-Aff^{\sim})$, which is determined by the natural isomorphisms

$$\mathbb{R}\underline{Hom}(F \times G, H) \simeq \mathbb{R}\underline{Hom}(F, \mathbb{R}\mathcal{HOM}(G,H)).$$

We say that $\mathbb{R}\mathcal{HOM}(F,G)$ is the *D-stack of morphisms* from F to G, analogous to the sheaf of morphisms between two sheaves.

If one looks at these various constructions, one realizes that $D-Aff^{\sim}$ has all the homotopy analogs of the properties that characterize Grothendieck topoi. To be more precise, C. Rezk has defined a notion of *homotopy topos* (we rather prefer the expression *model topos*), which are model categories behaving homotopically very much like a usual topos. The standard examples of such homotopy topoi are model categories of simplicial presheaves on some Grothendieck site, but not all of them are of this kind; the model category $D-Aff^{\sim}$ is in fact an example of a model topos which is not equivalent to model categries of simplicial presheaves on some site (see [HAG-I, §3.8] for more details on the subject).

5.4 First examples of D-stacks

Before going further with the geometric properties of D-stacks, we would like to present some examples. More examples will be given in the Section 5.

5.4.1 Representables

The very first examples of schemes are affine schemes. In the same way, our first example of D-stacks are *representable D-stacks*[5].

We start by fixing a fibrant resolution functor Γ on the model category $CDGA$. Recall that this means that for any cdga B, $\Gamma(B)$ is a simplicial object in $CDGA$, together with a natural morphism $B \longrightarrow \Gamma(B)$ that makes it into a fibrant replacement for the Reedy model structure on simplicial objects (see [Ho, §5.2]). In the present situation, one could choose the following standard fibrant resolution functor

$$\begin{array}{rccc} \Gamma(B): & \Delta^{op} & \longrightarrow & CDGA \\ & [n] & \mapsto & \Gamma(B)_n := B \otimes \Omega^*_{\Delta^n}. \end{array}$$

Here $\Omega^*_{\Delta^n}$ is the cdga (exceptionally positively graded) of algebraic differential forms on the standard algebraic n-simplex. Of course the cdga $B \otimes \Omega^*_{\Delta^n}$ is not non-positively graded, but one can always take its truncation in order to see it as an object in $CDGA$.

Now, for any cdga A, we define a functor

$$\begin{array}{rccc} Spec\,A: & CDGA & \longrightarrow & SSet \\ & B & \mapsto & Hom(A, \Gamma(B)), \end{array}$$

that is considered as an object in $D-Aff^{\sim}$. This construction is clearly functorial in A and gives rise to a functor

$$Spec: CDGA^{op} = D-Aff \longrightarrow D-Aff^{\sim}.$$

The functor *Spec* is almost a right Quillen functor: it preserves fibrations, trivial fibrations and limits, but does not have a left adjoint. However, it has a well defined right derived functor

$$\mathbb{R}Spec\ : \mathrm{Ho}(CDGA)^{op} = \mathrm{Ho}(D-Aff) \longrightarrow \mathrm{Ho}(D-Aff^{\sim}).$$

A fundamental property of this functor is the following lemma.

[5] We could as well have called them *affine D-stacks*.

Lemma 5.4.1 *The functor $\mathbb{R}Spec$ is fully faithful. More generally, for two cdga's A and B, it induces a natural equivalence on the mapping spaces*

$$\mathbb{R}\underline{Hom}(A,B) \simeq \mathbb{R}\underline{Hom}(\mathbb{R}Spec\, B, \mathbb{R}Spec\, A).$$

The above lemma contains two separated parts. The first part states that $\mathbb{R}Spec$ is fully faithful when considered to have values in $\mathrm{Ho}(D-Aff^{\wedge})$ (i.e. when one forgets about the topology). This first part is a very general result that we call *Yoneda lemma for model categories* (see [HAG-I, §4.2]). The second part of the lemma states that for a cofibrant cdga A, the object $Spec(A)$ is a D-stack (see Definition 5.3.10). This is not a general fact, and of course depends on the choice of the topology. Another way to express this last result is to say that the *étale topology is sub-canonical.*

Definition 5.4.2 *A D-stack isomorphic in* $\mathrm{Ho}(D-Aff^{\sim})$ *to some $\mathbb{R}Spec\, A$ is called a* representable *D-stack.*

In particular, Lemma 5.4.1 implies that the full subcategory of $\mathrm{Ho}(D-Aff)^{\sim}$ consisting of representable D-stacks is equivalent to the homotopy category of cdga's.

5.4.2 Stacks vs. D-stacks

Our second example of D-stacks are simply stacks. In other words, any stack defined over the category of affine schemes with the étale topology gives rise to a D-stack.

Let Alg be the category of commutative $\mathbb{C}$-algebras, and $Aff = Alg^{op}$ its opposite category. Recall that there exists a model category of simplicial presheaves on Aff for the étale topology (see [Ja1]). We will consider its projective version described in [Bl], and denote it by $Aff^{\sim}$. This model category is called the *model category of stacks for the étale topology.* Its homotopy category $\mathrm{Ho}(Aff^{\sim})$ contains as full subcategories the category of sheaves of sets and the category of stacks in groupoids (see e.g. [La-Mo]). More generally, one can show that the full subcategory of n-truncated objects in $\mathrm{Ho}(Aff^{\sim})$ is naturally equivalent to the homotopy category of stacks in n-groupoids (unfortunately there are no references for this last result until now but the reader might consult [Hol] for the case $n = 1$). In particular, $\mathrm{Ho}(Aff^{\sim})$ contains as a full subcategory the category of schemes, and more generally of Artin stacks.

There exists an adjunction

$$H^0 : CDGA \longrightarrow Alg \qquad CDGA \longleftarrow Alg : j,$$

for which j is the full embedding of Alg in $CDGA$ that sends a commutative algebra to the corresponding cdga concentrated in degree 0. Furthermore, this adjunction is a Quillen adjunction when Alg is endowed with its trivial model structure (as written above, j is on the right and H^0 is its left adjoint). This adjunction induces various adjunctions between the category of simplicial presheaves

$$j_! : Aff^{\sim} \longrightarrow D-Aff^{\sim} \qquad Aff^{\sim} \longleftarrow D-Aff^{\sim} : j^*$$

$$j^* : D-Aff^{\sim} \longrightarrow Aff^{\sim} \qquad D-Aff^{\sim} \longleftarrow Aff^{\sim} : (H^0)^*$$

One can check that these adjunction are Quillen adjunction (where the functors written on the left are left Quillen). In particular we conclude that j^* is right and left Quillen, and therefore preserves equivalences. ¿From this we deduce easily the following important fact.

Lemma 5.4.3 *The functor*

$$i := \mathbb{L}j_! : \mathrm{Ho}(Aff^{\sim}) \longrightarrow \mathrm{Ho}(D-Aff^{\sim})$$

is fully faithful.

The important consequence of the previous lemma is that $\mathrm{Ho}(D-Aff^{\sim})$ contains schemes, algebraic stacks ..., as full sub-categories.

Warning: The full embedding i does not commute with homotopy pull-backs, nor with internal Hom-D-stacks.

Here we reach the real heart of DAG: the category of D-stacks contains usual stacks, but these are not stable under the standard operations of D-stacks. In other words, if one starts with some schemes and performs some constructions on these schemes, considered as D-stacks, the result might not be a scheme anymore. This is the main reason why derived moduli spaces are not schemes, or stacks in general !

Notations. In order to avoid confusion, a scheme or a stack X, when considered as a D-stack will always be denoted by $i(X)$, or simply by iX.

The full emdedding $i = \mathbb{L}j_!$ has a right adjoint $\mathbb{R}j^* = j^*$. It will be denoted by

$$h^0 := j^* : \mathrm{Ho}(D-Aff^{\sim}) \longrightarrow \mathrm{Ho}(Aff^{\sim}),$$

and called the *truncation functor*. Note that for any cdga, one has

$$h^0(\mathbb{R}Spec\,A) \simeq Spec\,H^0(A),$$

which justifies the notation h^0. Note also that for any D-stack F, and any commutative algebra A, one has

$$F(A) \simeq \mathbb{R}\underline{Hom}(iSpec\,A, F) \simeq \mathbb{R}\underline{Hom}(Spec\,A, h^0(F)) \simeq h^0(F)(A).$$

This shows that a D-stack F and its truncation $h^0(F)$ *have the same points with values in commutative algebras.* Of course, F and $h^0(F)$ do not have the same points with values in cdga's in general, except when F is of the form iF' for some stack $F' \in \mathrm{Ho}(Aff^{\sim})$.

Terminology. Points with values in commutative algebras will be called *classical points.*

We just saw that a D-stack F and its truncation $h^0(F)$ always have the same classical points.

Given two stacks F and G in $Aff^{\sim}$, there exists a stack of morphisms $\mathbb{R}\mathcal{HOM}(F,G)$, that is the derived internal Hom's of the model category $Aff^{\sim}$ (see [HAG-I, §4.7]). As remarked above, the two objects $i\mathbb{R}\mathcal{HOM}(F,G)$ and $\mathbb{R}\mathcal{HOM}(iF,iG)$ are different in general. However, one has

$$h^0(\mathbb{R}\mathcal{HOM}(iF,iG)) \simeq \mathbb{R}\mathcal{HOM}(F,G),$$

showing that $i\mathbb{R}\mathcal{HOM}(F,G)$ and $\mathbb{R}\mathcal{HOM}(iF,iG)$ have the same classical points.

5.4.3 dg-Schemes

We have just seen that the homotopy category of D-stacks $\mathrm{Ho}(D-Aff^{\sim})$ contains the categories of schemes and algebraic stacks. We will now relate the notion of dg-schemes of [Ci-Ka1, Ci-Ka2] to D-stacks.

Recall that a dg-scheme is a pair $(X,\mathcal{A}_X)$, consisting of a scheme X and a sheaf of $\mathcal{O}_X$-cdga's on X such that $\mathcal{A}^0_X = \mathcal{O}_X$ (however, this last condition does not seem so crucial). For the sake of simplicity we will assume that X is quasi-compact and separated. We can therefore take a finite affine open covering $\mathcal{U} = \{U_i\}_i$ of X, and consider its nerve $N(\mathcal{U})$ (which is a simplicial scheme)

$$\begin{array}{cccc} N(\mathcal{U}): & \Delta^{op} & \longrightarrow & \{Schemes\} \\ & [n] & \mapsto & \coprod_{i_0,\dots,i_n} U_{i_0,\dots,i_n} \end{array}$$

where, as usual, $U_{i_0,\dots,i_n} = U_{i_0} \cap U_{i_1} \cap \dots U_{i_n}$. Note that as X is separated and the covering is finite, $N(\mathcal{U})$ is in fact a simplicial affine scheme.

For each integer n, let $A(n)$ be the cdga of global sections of $\mathcal{A}_X$ on the scheme $N(\mathcal{U})_n$. In other words, one has

$$A(n) = \prod_{i_0,\dots,i_n} \mathcal{A}_X(U_{i_0}) \times \mathcal{A}_X(U_{i_1}) \times \dots \mathcal{A}_X(U_{i_n}).$$

The simplicial structure on $N(\mathcal{U})$ makes $[n] \mapsto A(n)$ into a co-simplicial diagram of cdga's. By applying levelwise the functor $\mathbb{R}Spec$, we get a simplicial object $[n] \mapsto \mathbb{R}Spec\, A(n)$ in $D - Aff^{\sim}$. We define the D-stack $\Theta(X, \mathcal{A}_X) \in \mathrm{Ho}(D - Aff^{\sim})$ to be the homotopy colimit of this diagram

$$\Theta(X, \mathcal{A}_X) := Hocolim_{[n]\in\Delta^{op}} \mathbb{R}Spec\, A(n).$$

One can check, that $(X, \mathcal{A}_X) \mapsto \Theta(X, \mathcal{A}_X)$ defines a functor

$$\Theta : \mathrm{Ho}(dg - Sch) \longrightarrow \mathrm{Ho}(D - Aff^{\sim}),$$

from the homotopy category of (quasi-compact and separated) dg-schemes to the homotopy category of D-stacks. This functor allows us to consider dg-schemes as D-stacks.

Question: *Is the functor Θ fully faithful ?*

We do not know the answer to this question, and there are no real reasons for this answer to be positive. As already explained in the Introduction, the difficulty comes from the fact that the homotopy category of dg-schemes seems quite difficult to describe. In a way, it might not be so important to know the answer to the above question, as until now morphisms in the homotopy category of dg-schemes have never been taken into account seriously, and only the objects of the category $\mathrm{Ho}(dg - Sch)$ have been shown to be relevant. More fundamental is the existence of the functor Θ which allows to see the various dg-schemes constructed in [Ka2, Ci-Ka1, Ci-Ka2] as objects in $\mathrm{Ho}(D - Aff^{\sim})$.

Remark 5.4.4 The above construction of Θ can be extended from dg-schemes to (Artin) dg-stacks.

5.4.4 The D-stack of G-torsors

As our last example, we present the D-stack of G-torsors where G is a linear algebraic group G. As an object in $\mathrm{Ho}(D - Aff^{\sim})$ it is simply iBG (where BG is the usual stack of G-torsors), but we would like to describe explicitly the functor $CDGA \longrightarrow SSet$ it represents.

Let $H := \mathcal{O}(G)$ be the Hopf algebra associated to G. By considering it as an object in the model category of commutative differential graded Hopf algebras, we can take a cofibrant model QH of H, as a dg-Hopf algebra. It is not very hard to check that QH is also a cofibrant model for H is the model category of cdga's. Using the co-algebra structure on QH, one sees that the simplicial presheaf

$$Spec\, QH : D - Aff^{op} \longrightarrow SSet$$

has a natural structure of group-like object. In other words, $Spec\, QH$ is a presheaf of simplicial groups on $D - Aff$. As the underlying simplicial presheaf of $Spec\, QH$ is naturally equivalent to $\mathbb{R}Spec\, H \simeq iG$, we will simply denote this presheaf of simplicial groups by iG.

Next, we consider the category $iG - Mod$, of objects in $D - Aff^{\sim}$ together with an action of iG. If one sees iG as a monoid in $D - Aff^{\sim}$, the category $iG - Mod$ is simply the category of modules over iG. The category $iG - Mod$ is equipped with a notion of weak equivalences, that are defined through the forgetful functor $iG - Mod \longrightarrow D - Aff^{\sim}$ (therefore a morphism of iG-modules is a weak equivalence if the morphism induced on the underlying objects is a weak equivalence in $D - Aff^{\sim}$). More generally, there is a model category structure on $iG - Mod$, such that fibrations and equivalences are defined on the underlying objects. For any object $F \in iG - Mod$, we also get an induced model structure on the comma category $iG - Mod/F$. In particular, it makes sense to say that two objects $G \longrightarrow F$ and $G' \longrightarrow F$ in $iG - Mod$ are equivalent over F, if the corresponding objects in $\mathrm{Ho}(iG - Mod/F)$ are isomorphic.

Let Q be a cofibrant replacement functor in the model category $CDGA$. For any cdga A, we have $Spec\,QA \in D - Aff^{\sim}$, the representable D-stack represented by $A \in D - Aff$, that we will consider as iG-module for the trivial action. A *G-torsor over A* is defined to be a iG-module $F \in iG - Mod$, together with a fibration of iG-modules $F \longrightarrow Spec\,QA$, such that there exists an étale covering $A \longrightarrow B$ with the property that the object

$$F \times_{Spec\,QA} Spec\,QB \longrightarrow Spec\,QB$$

is equivalent over $Spec\,QB$ to $iG \times Spec\,QB \longrightarrow Spec\,QB$ (where iG acts on itself by left translations).

For a cdga A, G-torsors over A form a full sub-category of $iG - Mod/Spec\,QA$, that will be denoted by $G - Tors(A)$. This category has an obvious induced notion of weak equivalences, and these equivalences form a subcategory denoted by $wG - Tors(A)$. Transition morphisms $wG - Tors(A) \longrightarrow wG - Tors(B)$ can be defined for any morphism $A \longrightarrow B$ by sending a G-torsor $F \longrightarrow Spec\,QA$ to the pull-back $F \times_{Spec\,QA} Spec\,QB \longrightarrow Spec\,QB$. With a bit of care, one can make this construction into a (strict) functor

$$\begin{array}{ccc} CDGA & \longrightarrow & Cat \\ A & \mapsto & wG - Tors(A). \end{array}$$

We are now ready to define our functor

$$\begin{array}{cccc} \mathbb{R}BG : & CDGA & \longrightarrow & SSet \\ & A & \mapsto & |wG - Tors(A)|, \end{array}$$

where $|wG - Tors(A)|$ is the nerve of the category $wG - Tors(A)$. The following result says that $\mathbb{R}BG$ is the *associated D-stack to iBG* (recall that BG is the Artin stack of G-torsors, and that iBG is its associated D-stack defined through the embedding i of Lemma 5.4.3).

Proposition 5.4.5 *1. The object $\mathbb{R}BG \in D - Aff^{\sim}$ is a D-stack.*

2. There exists an isomorphism $iBG \simeq \mathbb{R}BG$ in the homotopy category $\mathrm{Ho}(D - Aff^{\sim})$.

An important case is $G = Gl_n$, for which we get that the image under i of the stack $\underline{Vect}_n$ of vector bundles of rank n is equivalent to $\mathbb{R}BGl_n$ as defined above.

5.5 The geometry of D-stacks

We are now ready to start our geometric study of D-stacks. We will define in this Section a notion of (1)-*geometric* D-stack, analogous to the notion of algebraic stack (in the sense of Artin). We will also present the theory of *tangent D-stacks*, as well as its relations to the cotangent complex.

5.5.1 Geometricity

A 1-geometric D-stack is a *quotient of disjoint union of representable D-stacks by the action of a smooth affine groupoid.* In order to define precisely this notion, we need some preliminaries.

1. Let $f : F \longrightarrow F'$ be a morphism in $\mathrm{Ho}(D - Aff^{\sim})$. We say that f is a *representable morphism*, if for any cgda A, and any morphism $\mathbb{R}Spec\, A \longrightarrow F'$, the homotopy pull-back $F \times^h_{F'} \mathbb{R}Spec\, A$ is a representable D-stack (see Definition 5.4.2).
2. We say that a D-stack F has a representable diagonal if the diagonal morphism $\Delta : F \longrightarrow F \times F$ is representable. Equivalently, F has a representable diagonal if any morphism $\mathbb{R}Spec\, A \longrightarrow F$ from a representable D-stack is a representable morphism.
3. Let $u : A \longrightarrow B$ be a morphism of cdga's. We say that u is *strongly smooth*[6] if there exists an étale covering $B \longrightarrow B'$, and a factorization

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \otimes \mathbb{C}[X_1, \ldots, X_n] & \longrightarrow & B' \end{array}$$

   with $A \otimes \mathbb{C}[X_1, \ldots, X_n] \longrightarrow B'$ formally étale; here $\mathbb{C}[X_1, \ldots, X_n]$ is the usual polynomial ring, viewed as a cdga concentrated in degree zero. This is an extension of one of the many equivalent characterizations of smoothness for morphisms of schemes (see [Mil, Prop. 3.24 (b)]); we learn it from [MCM] in which smooth morphisms (called there *thh-smooth*) between $\mathbb{S}$-algebras are defined.
4. A representable morphism of D-stacks $f : F \longrightarrow F'$ is called *strongly smooth*, if for any morphism from a representable D-stack $\mathbb{R}Spec\, A \longrightarrow F'$, the induced morphism

$$F \times^h_{F'} \mathbb{R}Spec\, A \longrightarrow \mathbb{R}Spec\, A$$

   is induced by a strongly smooth morphism of cdga's.
5. A morphism $f : F \longrightarrow F'$ in $\mathrm{Ho}(D - Aff^{\sim})$ is called a *covering* (or an epimorphism), if the induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is an epimorphism of sheaves.

Note that definition (4) above makes sense because of (1) and because the functor $A \mapsto \mathbb{R}Spec\, A$ is fully faithful on the homotopy categories.

Using these notions, we give the following

Definition 5.5.1 *A D-stack F is* strongly (1)-geometric *if it satisfies the following two conditions*

1. *F has a representable diagonal.*
2. *There exist representable D-stacks $\mathbb{R}Spec\, A_i$, and a covering*

$$\coprod_i \mathbb{R}Spec\, A_i \longrightarrow F,$$

   such that each of the morphisms $\mathbb{R}Spec\, A_i \longrightarrow F$ (which is representable by 1.) is strongly smooth. Such a family of morphisms will be called a strongly smooth atlas of F.

[6]The expression *smooth morphism* will be used for a weaker notion in §4.4.

Remark 5.5.2 Objects satisfying Definition 5.5.1 are called strongly 1-geometric D-stacks as there exists a more general notion of strongly n-geometric D-stacks, obtained by induction as suggested in [S2]. The notion of strongly 1-geometric D-stacks will be enough for our purposes (except for our last example in section 5), and we will simply use the expression *strongly geometric D-stacks*.

The following proposition collects some of the basic properties of strongly geometric D-stacks.

Proposition 5.5.3
1. *Representable D-stacks are strongly geometric.*
2. *The homotopy pull-back of a diagram of strongly geometric D-stacks is again a strongly geometric D-stack. In particular strongly geometric D-stacks are stable by finite homotopy limits.*
3. *If F is any algebraic stack (in the sense of Artin, see [La-Mo]) with an affine diagonal, then iF is a strongly geometric D-stack.*
4. *If F is a strongly geometric D-stack then $h^0(F)$ is an algebraic stack (in the sense of Artin) with affine diagonal. In particular, $ih^0(F)$ is again a strongly geometric D-stack.*
5. *For any dg-scheme $(X, \mathcal{A}_X)$, (X separated and quasi-compact), $\Theta(X, \mathcal{A}_X)$ (see §3.3) is a strongly geometric D-stack.*

In particular, Proposition 5.4.5 and point (3) above, tell us that the derived stack $\mathbb{R}BG$ of G-torsors is a stongly geometric D-stack for any linear algebraic group G.

We are not going to present the theory in details in this work, but we would like to mention that standard notions in algebraic geometry (e.g. smooth or flat morphisms, sheaves, cohomology ...) can be extended to strongly geometric D stacks. We refer to [La-Mo] and [S2] for the main outline of the constructions. The reader will find all details in [HAG-II].

5.5.2 Modules, linear D-stacks and K-theory

Let $\mathbb{G}_a$ be the additive group scheme (over $\mathbb{C}$) and consider the object $i\mathbb{G}_a \in \mathrm{Ho}(D - Aff^{\sim})$. It has a nice model in $D - Aff^{\sim}$ which is $Spec\, \mathbb{C}[T]$ that we will denote by $\mathcal{O}$ (note that $\mathbb{C}[T]$ as a cdga in degree 0 is a cofibrant object). The D-stack $\mathcal{O}$ is actually an object in commutative $\mathbb{C}$-algebras, explicitly given by

$$\begin{array}{rccc} \mathcal{O}: & CDGA & \longrightarrow & (\mathbb{C} - Alg)^{\Delta^{op}} \\ & A & \mapsto & ([n] \mapsto \Gamma_n(A)^0), \end{array}$$

where Γ is a fibrant resolution functor. The D-stack is called the *structural D-stack*.

Let us now fix a D-stack F, and consider the comma category $D - Aff^{\sim}/F$ of D-stacks over F; this category is again a model category for the obvious model structure. We define the *relative structural D-stack* by

$$\mathcal{O}_F := \mathcal{O} \times F \longrightarrow F \in D - Aff^{\sim}/F.$$

Since $\mathcal{O}$ is a $\mathbb{C}$-algebra object, we deduce immediately that $\mathcal{O}_F$ is also a $\mathbb{C}$-algebra object in the comma model category $D - Aff^{\sim}/F$.

Then we can consider the category $\mathcal{O}_F - Mod$, of objects in $\mathcal{O}_F$-modules in the category $D - Aff^{\sim}/F$. If one defines equivalences and fibrations through the forgetful functor $D - Aff^{\sim}/F \longrightarrow D - Aff^{\sim}$, the category $\mathcal{O}_F - Mod$ becomes a model category. It has moreover a natural tensor product structure $\otimes_{\mathcal{O}_F}$. The model category $\mathcal{O}_F - Mod$ is called the *model category of $\mathcal{O}$-modules on F*.

Let A be a cdga and M be an (unbounded) A-dg module. We define a $\mathcal{O}_{Spec\, A}$-module $\widetilde{M}$ in the following way.

Let Γ be a fibrant resolution functor on the model category $CDGA$. For any cdga B, and any integer n, we define $\widetilde{M}(B)_n$ as the set of pairs (u,m), where u is a morphism of cdga's $A \longrightarrow \Gamma_n(B)$ (i.e. $u \in Spec\,A(B)$), and m is a degree 0 element in $M \otimes_A \Gamma_n(B)$ (i.e. m is a morphism of complexes of $\mathbb{C}$-vector spaces $m : \mathbb{C} \longrightarrow M \otimes_A \Gamma_n(B)$). This gives a simplicial set $[n] \mapsto \widetilde{M}(B)_n$, and therefore defines an object in $D - Aff^{\sim}$

$$\begin{array}{rccc} \widetilde{M}: & CDGA & \longrightarrow & SSet \\ & B & \mapsto & \widetilde{M}(B). \end{array}$$

Clearly, the projection $(u,m) \mapsto u$ in the notation above induces a morphism $\widetilde{M} \longrightarrow Spec\,A$. Finally, this object is endowed in an obvious way with a structure of $\mathcal{O}_{Spec\,A}$-module.

This construction, $M \mapsto \widetilde{M}$ induces a functor

$$\widetilde{M} : A - Mod \longrightarrow \mathcal{O}_{Spec\,A} - Mod$$

from the category of (unbounded) dg-A-modules, to the category of $\mathcal{O}_{Spec\,A}$-modules. This functor can be derived (by taking first cofibrant replacements of both A and M) to a functor

$$\mathbb{R}\widetilde{M} : \mathrm{Ho}(A - Mod) \longrightarrow \mathrm{Ho}(\mathcal{O}_{\mathbb{R}Spec\,A} - Mod).$$

Lemma 5.5.4 *The functor $\mathbb{R}\widetilde{M}$ defined above is fully faithful.*

Definition 5.5.5
1. *A $\mathcal{O}$-module on a representable D-stack $\mathbb{R}Spec\,A$ is called* pseudo-quasi-coherent *if it is equivalent to some $\mathbb{R}\widetilde{M}$ as above.*
2. *Let F be a D-stack, and $\mathcal{M}$ be a $\mathcal{O}$-module. We say that $\mathcal{M}$ is* pseudo-quasi-coherent *if for any morphism $u : \mathbb{R}Spec\,A \longrightarrow F$, the pull-back $u^*\mathcal{M}$ is a quasi-pseudo-coherent $\mathcal{O}$-module on $\mathbb{R}Spec\,A$.*

The construction $M \mapsto \widetilde{M}$ described above also has a dual version, denoted by $M \mapsto Spel(M)$ and defined in a similar way.

Let A be a cdga and M be an (unbounded) dg-A-module. For a cdga B and an integer n, we define $Spel(M)(B)_n$ to be the set of pairs (u,α), where $u : A \longrightarrow \Gamma_n(B)$ is a morphism of cdga, and $\alpha : M \longrightarrow \Gamma_n(B)$ is a morphism of dg-A-modules. This defines a D-stack $B \mapsto Spel(M)(B)$ which has a natural projection $(u,\alpha) \mapsto u$, to the $Spec\,A$. Once again, $Spel(M)$ comes equipped with a natural structure of $\mathcal{O}_{Spec\,A}$-module. Also, this $Spel$ construction can be derived, to get a functor

$$\mathbb{R}Spel : \mathrm{Ho}(A - Mod)^{op} \longrightarrow \mathrm{Ho}(\mathcal{O}_{\mathbb{R}Spec\,A} - Mod).$$

Lemma 5.5.6 *The functor $\mathbb{R}Spel$ defined above is fully faithful.*

Definition 5.5.7
1. *A $\mathcal{O}$-module on a representable D-stack $\mathbb{R}Spec\,A$ is called* representable *if it is equivalent to some $\mathbb{R}Spel(M)$ as above.*
2. *Let F be a D-stack, and $\mathcal{M}$ be a $\mathcal{O}$-module. We say that $\mathcal{M}$ is* representable *or is a* linear D-stack over F *if for any morphism $u : \mathbb{R}Spec\,A \longrightarrow F$, the pull-back $u^*\mathcal{M}$ is a representable $\mathcal{O}$-module on $\mathbb{R}Spec\,A$.*
3. *A* perfect $\mathcal{O}$-module *on a D-stack F is a $\mathcal{O}_F$-module which is both pseudo-quasi-coherent and representable.*

One can prove that the homotopy category of perfect $\mathcal{O}$-modules on $\mathbb{R}Spec\,A$ is naturally equivalent to the full sub-category of $\mathrm{Ho}(A - Mod)$ consisting of strongly dualizable modules, or equivalently of dg-A-modules which are retracts of finite cell modules (in the sense of [Kr-Ma, §III.1]). In particular, if A is concentrated in degree 0, then the homotopy category of perfect $\mathcal{O}$-modules on $\mathbb{R}Spec\, A$ is naturally equivalent to the derived category of bounded complexes of finitely generated projective A-modules.

This notion of perfect $\mathcal{O}$-modules can be used in order to define the *K-theory of D-stacks*. For any D-stack F, one can consider the homotopy category of perfect $\mathcal{O}$-modules on F, that we denote by $D_{\mathrm{Perf}}(F)$. This is a triangulated category having a natural Waldhausen model $W\mathrm{Perf}(F)$, from which one can define the K-theory spectra on the D-stack F, as $K(F) := K(W\mathrm{Perf}(F))$. The tensor product of $\mathcal{O}$-modules makes $K(F)$ into an E_∞-ring spectra. Of course, when X is a scheme $K(iX)$ is naturally equivalent to the K-theory spectra of X as defined in [TT].

A related problem is that of defining reasonable *Chow groups* and *Chow rings* for strongly geometric D-stacks, receiving Chern classes from the K-theory defined above. We are not aware of any such constructions nor we have any suggestion on how to approach the question. It seems however that an *intersection theory over* D-stacks would be a very interesting tool, as it might for example give new interpretations (and probably extensions) of the notion of virtual fundamental class defined in [Be-Fa]. For this case, the idea would be that for any strongly geometric D-stack F, there exists a virtual fundamental class in the Chow group of its truncation h^0F. The structural sheaf of F should give rise, in the usual way, to a fundamental class in its Chow group, such that integrating against it over the all F is the same thing as integrating on its truncation h^0F against the virtual fundamental class. However, even if there is still no theory of Chow groups for D-stacks, if one is satisfied with working with K-theory instead of Chow groups, the obvious class $1 =: [\mathcal{O}_F] \in K_0(F)$, will correspond exactly to the class of the expected virtual structure sheaf.

5.5.3 Tangent D-stacks

Let $Spec\,\mathbb{C}[\epsilon]$ the spectrum of the dual numbers, and let us consider $iSpec\,\mathbb{C}[\epsilon] \in \mathrm{Ho}(D-Aff^{\sim})$.

Definition 5.5.8 *The* tangent D-stack *of a D-stack F is defined to be*

$$\mathbb{R}TF := \mathbb{R}\mathcal{HOM}(iSpec\,\mathbb{C}[\epsilon], F) \in \mathrm{Ho}(D-Aff^{\sim}).$$

Note that the zero section morphism $Spec\,\mathbb{C} \longrightarrow Spec\,\mathbb{C}[\epsilon]$ and the natural projection $Spec\,\mathbb{C}[\epsilon] \longrightarrow Spec\,\mathbb{C}$ induces natural morphisms

$$\pi : \mathbb{R}TF \longrightarrow F \qquad e : F \longrightarrow \mathbb{R}TF,$$

where e is a section of π.

An important remark is that for any D-stack F, the truncation $h^0\mathbb{R}TF$ is equivalent to the tangent stack of h^0F (in the sense of [La-Mo, §17]). In other words, one has

$$h^0\mathbb{R}TF \simeq T(h^0F).$$

In particular, the D-stacks $\mathbb{R}TF$ and $iT(h^0F)$ have the same classical points. However, it is *not* true in general that $iTF \simeq \mathbb{R}T(iF)$ for a stack F. Even for a scheme X, it is not true that $\mathbb{R}T(iX) \simeq iTX$, except when X is smooth.

Definition 5.5.9 *If $x : iSpec\,\mathbb{C} \longrightarrow F$ is a point of a D-stack F, then the* tangent D-stack of F at x *is the homotopy fiber of $\pi : \mathbb{R}TF \longrightarrow F$ at the point x. It is denoted by*

$$\mathbb{R}TF_x := \mathbb{R}TF \times^h_F iSpec\,\mathbb{C} \in \mathrm{Ho}(D-Aff^{\sim}).$$

Let us now suppose that F is a strongly geometric D-stack. One can show that $\mathbb{R}TF$ is also strongly geometric. In particular, for any point x in $F(\mathbb{C})$ the D-stack $\mathbb{R}TF_x$ is strongly geometric.

Actually much more is true. For any strongly geometric D-stack F, and any point x in $F(\mathbb{C})$, the D-stack $\mathbb{R}TF_x$ is a *linear D-stack* (over $i\mathrm{Spec}\mathbb{C}$) as defined in 5.5.7. Let us recall that this implies the existence of a natural complex $\mathbb{R}\Omega^1_{F,x}$ of $\mathbb{C}$-vector spaces (well defined up to a quasi-isomorphism and concentrated in degree $]-\infty,1]$), with the property that, for any cdga A, there exists a natural equivalence

$$\mathbb{R}TF_x(A) \simeq \mathbb{R}\underline{Hom}_{C(\mathbb{C})}(\mathbb{R}\Omega^1_{F,x}, A),$$

where $\mathbb{R}\underline{Hom}_{C(\mathbb{C})}$ denotes the mapping space in the model category of (unbounded) complexes of $\mathbb{C}$-vector spaces. Symbolically, one writes

$$\mathbb{R}TF_x = (\mathbb{R}\Omega^1_{F,x})^*,$$

where $(\mathbb{R}\Omega^1_{F,x})^*$ is the dual complex to $\mathbb{R}\Omega^1_{F,x}$. In other words, the tangent D-stack of F at x “is” the complex $(\mathbb{R}\Omega^1_{F,x})^*$, which is now concentrated in degree $[-1,\infty[$.

Definition 5.5.10 *If $x : iSpec\mathbb{C} \longrightarrow F$ is a point of a strongly geometric D-stack, then we say that the* dimension of F at x is defined *if the complex $\mathbb{R}\Omega^1_{F,x}$ has bounded and finite dimensional cohomology. If this is the case, the* dimension of F at x *is defined by*

$$\mathbb{R}Dim_x F := \sum_i (-1)^i H^i(\mathbb{R}\Omega^1_{F,x}).$$

This linear description of $\mathbb{R}TF_x$ has actually a global version. In fact, one can define a *cotangent complex* $\mathbb{R}\Omega^1_F$ of a strongly geometric D-stack, which is in general an $\mathcal{O}$-module on F in the sense of Definition 5.5.5, which is most of the times quasi-coherent. One then shows that there exists an equivalence of D-stacks over F

$$\mathbb{R}TF \simeq \mathbb{R}Spel\,(\mathbb{R}\Omega^1_F),$$

and in particular that the D-stack $\mathbb{R}TF$ is a *linear stack* over F in the sense of Definition 5.5.7.

An already interesting application of this description, is to the case $F = iX$, for X a scheme or even an algebraic stack. Indeed, the cotangent complex $\mathbb{R}\Omega^1_{iX}$ mentioned above is precisely the cotangent complex $\mathbb{L}_X$ of [La-Mo, §17]. The equivalence

$$\mathbb{R}T(iX) \simeq \mathbb{R}Spel\,(\mathbb{R}\Omega^1_X)$$

gives a relation between the *purely algebraic* object $\mathbb{L}_X$ and the *geometric object* $\mathbb{R}T(iX)$. In a sense, the usual geometric intuition about the tangent space is recovered here, at the price of (and thanks to) enlarging the category of objects under study: the cotangent complex of a scheme becomes the derived tangent space of the scheme considered as a D-stack. We like to see this as a possible answer to the following remark of A. Grothendieck ([Gr1, p. 4]):

[...] *Il est très probable que cette théorie pourra s'étendre de façon à donner une correspondance entre complexes de chaines de longeur n, et certaines “n-catégories” cofibrées sur* $\underline{C}$*; et il n'est pas exclus que par cette voie on arrivera également à une “interprétation géométrique” du complexe cotangent relatif de Quillen.*

5.5.4 Smoothness

To finish this part, we investigate various non-equivalent natural notions of smoothness for geometric D-stacks.

Strong smoothness. We have already defined the notion of a strongly smooth morphisms of cdga's in §4.1. We will therefore say that a morphism

$$F \longrightarrow \mathbb{R}Spec\, B$$

from a geometric D-stack F is *strongly smooth* if there is a strongly smooth atlas $\coprod \mathbb{R}Spec\, A_i \longrightarrow F$ as in Definition 5.5.1, such that all the induced morphisms of cdga's $B \longrightarrow A_i$ are strongly smooth morphisms of cdga's (see §4.1). More generally, a morphism between strongly geometric D-stacks, $F \longrightarrow F'$, is called strongly smooth if for any morphism $\mathbb{R}Spec\, B \longrightarrow F'$ the morphism $F \times^h_{F'} \mathbb{R}Spec\, B \longrightarrow \mathbb{R}Spec\, B$ is strongly smooth in the sense above.

Strong smoothness is not very interesting for D-stacks, as a strongly geometric D-stack F will be strongly smooth if and only if it is of the form iF', for F' a *smooth* algebraic stack.

Standard smoothness. A more interesting notion is that of *standard smooth morphisms*, or simply *smooth morphisms*. On the level of cdga's they are defined as follows.

A morphism of cdga's $A \longrightarrow B$ is called *standard smooth* (or simply *smooth*), if there exists an étale covering $B \longrightarrow B'$, and a factorization

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B', \end{array}$$

such that the A-algebra A' is equivalent to a free A-algebra over some perfect dg-A-module (i.e. $A \longrightarrow A'$ is free and finitely presented). This notion, defined on cdga's, can be extended (as we did above for strongly smooth morphisms) to morphisms between strongly geometric D-stacks.

This notion is more interesting than strong smoothness, as a strongly geometric D-stack can be smooth without being an algebraic stack. However, one can check that if F is a smooth strongly geometric D-stack in this sense, then $h^0(F)$ is also a smooth algebraic stack. In particular, the derived version of the stack of vector bundles on a smooth projective surface, discussed in the Introduction (see also conjecture 5.6.4), will never be smooth in this sense as its truncation is the stack of vector bundles on the syrface which is singular in general).

Nevertheless, smooth morphisms can be used in order to define the following more general notion of geometric D-stacks.

Definition 5.5.11 *A D-stack F is* (1)-geometric *if it satisfies the following two conditions*

1. *The D-stack has a representable diagonal.*
2. *There exists representable D-stacks $\mathbb{R}Spec\, A_i$, and a covering*

$$\coprod_i \mathbb{R}Spec\, A_i \longrightarrow F,$$

such that each of the morphisms $\mathbb{R}Spec\, A_i \longrightarrow F$ is smooth. Such a family of morphisms will be called a smooth atlas *of F.*

Essentially all what we have said about strongly geometric D-stacks is also valid for geometric D-stacks in the above sense. In Section 5 we will give some natural examples of geometric D-stacks which are not strongly geometric.

fp-smoothness. The third notion of smoothness is called *fp-smoothness* and is the weakest of the three and it seems this is the one which is closer to the smoothness notion referred to in the DDT program in general. It is also well suited in order for the derived stack of vector bundles to be smooth.

Recall that a morphism of cgda's, $A \longrightarrow B$ is *finitely presented* if it is equivalent to a retract of a finite cell A-algebra, or equivalently if the mapping space $\mathrm{Map}_{A/CDGA}(B,-)$ commutes with filtered colimits (this is the same as saying that $\mathbb{R}Spec\, A$ commutes with filtered colimits). We will then say that a morphism of geometric D-stacks, $F \longrightarrow F'$ is *locally finitely presented* if for any morphism $\mathbb{R}Spec\, A \longrightarrow F'$ there exists a smooth atlas

$$\coprod \mathbb{R}Spec\, A_i \longrightarrow F \times_{F'} \mathbb{R}Spec\, A$$

such that all the induced moprhisms of cdga's $A \longrightarrow A_i$ are finitely presented. Locally finitely presented morphisms will also be called *fp-smooth* morphisms. The reason for this name is given by the following observation.

Proposition 5.5.12 *Le F be a geometric D-stack which is fp-smooth (i.e. $F \longrightarrow * = i\mathrm{Spec}\mathbb{C}$ is fp-smooth). Then the cotangent complex $\mathbb{R}\Omega^1_F$ is a perfect complex of $\mathcal{O}$-modules on F.*

In particular, for any point $x \in F(\mathbb{C})$, the dimension of F at x is defined and locally constant for the étale topology.

Of course, one has *strongly smooth* $\Rightarrow$ *smooth* $\Rightarrow$ *fp-smooth*, but each of these implications is strict. For example, a smooth scheme is strongly smooth. Let E be a complex in non-positive degrees which is cohomologically bounded and of finite dimension. Then $\mathbb{R}Spel(E)$ is smooth but not strongly smooth as it is not a scheme in general. Finally, any scheme which is a local complete interesection is fp-smooth, but not smooth in general.

5.6 Further examples

In this Section we present three examples of geometric D-stacks: the derived stack of *local systems on a space*, the derived stack of *vector bundles* and the derived stack of *associative algebra* and A_∞*-categorical structures*. The derived moduli space of local systems on a space has already been introduced and defined in [Ka2] as a dg-scheme. In the same way, the derived moduli space of (commutative) algebra structures has been constructed in [Ci-Ka2] also as a dg-scheme. Finally, the formal derived moduli spaces of local systems on a space and of A_∞-categorical structures have been considered in [Hin2, Ko2, Ko-So].

The new mathematical content of this part is the following. First of all we give a construction of the derived moduli stack of vector bundles, that seems to be new, and we also define global versions of the formal moduli spaces of A_∞-categorical structures that were apparently not known. We also provide explicit modular descriptions, by defining various derived moduli functors, which were not known (and probably not easily available), for the constructions of [Ka2, Ci-Ka1, Ci-Ka2].

5.6.1 Local systems on a topological space

Throughout this subsection, X will be a CW-complex. For any cdga A, we denote by $A - Mod_X$ the category of presheaves of dg-A-modules over X. We say that a map $\mathcal{M} \to \mathcal{N}$ in $A - Mod_X$ is a *quasi-isomorphism* if it induces a quasi-isomorphism of dg-A-modules on each stalk. This gives a notion of *equivalences* in the category $A - Mod_X$, and of *equivalent objects* (i.e. objects which are isomorphic in the localization of the category with respect to equivalences).

A presheaf $\mathcal{M}$ of dg-A-modules on X will be said *locally on $X \times A_{\text{ét}}$ equivalent to A^n* if, for any $x \in X$, there exists an open neighborhood U of x in X and an étale cover $A \to B$, such that the presheaves of dg-B-modules $\mathcal{M}_{|U} \otimes_A B$ and B^n are equivalent. We will also say that a presheaf $\mathcal{M}$ of dg-A-modules is *flat*, if for any open U of X, the dg-A-module $\mathcal{M}(U)$ is cofibrant. By composing with a cofibrant replacement functor in $A - Mod$, one can associate to any dg-A-module an equivalent flat dg-A-module (since equivalences are stable by filtered colimits). The category $w\text{Loc}_n(X;A)$ *of rank n local systems of dg-A-modules* has objects those presheaves of flat dg-A-modules on X which are locally on $X \times A_{\text{ét}}$ equivalent to A^n, and morphisms quasi-isomorphisms between them. For morphisms of cdga's $A \longrightarrow B$ we obtain pull-back functors

$$\begin{array}{ccc} w\text{Loc}_n(X;A) & \longrightarrow & w\text{Loc}_n(X;B) \\ \mathcal{M} & \mapsto & \mathcal{M} \otimes_A B. \end{array}$$

This makes $wLoc_n(X;A)$ into a lax functor from $CDGA$ to categories, that we turn into a strict functor by applying the standard strictification procedure.

We denote by $\mathbb{R}\underline{Loc}_n(X)$ the simplicial presheaf on $D-Aff$ sending a cdga A to $|w\text{Loc}_n(X)|$ (the nerve of $w\text{Loc}_n(X;A)$). We call it the *D-pre-stack of rank n derived local systems on X.*

Obviously, the objects in $w\text{Loc}_n(X;A)$ are a derived version of the usual local systems of R-modules on X, where R is a commutative ring. More precisely, if we consider such an R as a cdga concentrated in degree zero, then $R\text{Loc}_n(X;R)$ is the closure under quasi-isomorphisms of the groupoid of rank n local systems of R-modules on X; in other words, if we invert quasi-isomorphisms in the category $w\text{Loc}_n(X;R)$ then we obtain a category which is equivalent to the groupoid of rank n local systems of R-modules on X.

Theorem 5.6.1 *1. The D-pre-stack $\mathbb{R}\underline{Loc}_n(X)$ is a D-stack. Furthermore, one has $\mathbb{R}\underline{Loc}_n(pt) \simeq iBGl_n$.*

2. One has an equivalence

$$h^0\mathbb{R}\underline{Loc}_n(X) \simeq [Hom(\pi_1(X), Gl_n)/Gl_n],$$

between the truncation of $\mathbb{R}\underline{Loc}_n(X)$ and the (Artin) stack of local systems on X.

3. If $S(X)$ denotes the singular complex of X, we have the following isomorphisms in $\text{Ho}(D-Aff^{\sim})$,

$$\mathbb{R}\underline{Loc}_n(X) \simeq \mathbb{R}\mathcal{HOM}(\underline{S(X)}, iBGl_n) \simeq \mathbb{R}\mathcal{HOM}(\underline{S(X)}, \text{Loc}_n(\text{pt})),$$

where $\mathbb{R}\mathcal{HOM}$ denotes the Hom-stack (internal Hom in $\text{Ho}(D-Aff^{\sim})$*) and $\underline{S(X)}$ denotes the simplicial constant presheaf with value $S(X)$.*

4. For any rank n local system L on X, the tangent D-stack of $\mathbb{R}\underline{Loc}_n(X)$ at L is the complex $C^*(X, \underline{End}(L))[1]$, of cohomology of X with coefficients in $\underline{End}(L)$.

5. If X is a finite *CW-complex, then the stack $\mathbb{R}\underline{Loc}_n(X)$ is strongly geometric, fp-smooth of (the expected) dimension $-n^2\chi(X)$, $\chi(X)$ being the Euler characteristic of X.*

Note that the classical points of $\mathbb{R}\underline{Loc}_n(X)$ (i.e. morphisms from $i\text{Spec}k$, for some commutative ring k) coincide with the classical points of its truncation $h^0\mathbb{R}\underline{Loc}_n(X)$ which coincides with the usual (i.e. not derived) stack of rank n local systems on X. So we have no new classical points, as desired.

Let us give only some remarks to show what the proof of Theorem 5.6.1 really boils down to. First of all notice that the first assertion is a consequence of the second one, once one knows that $\mathbb{R}\underline{Loc}_n(\text{pt}) \simeq iBGL_n$ and is a stack; so we are reduced to prove the absolute case ($X = \text{pt}$) of 1. and 2. The first two properties in 3. follows from 2., the finiteness of X and the analogous properties of BGl_n. Finally the dimension count in 3. is made by an explicit computaion of the tangent D-stack at some local system E. Explicitly, one finds that (in the notations of §4.3) $(\mathbb{R}\Omega^1_{\mathbb{R}\underline{Loc}_n(X),E})^*$ is the complex $C^*(X, \underline{End}(E))[1]$, which is a complex of $\mathbb{C}$-vector spaces concentrated in degrees $[-1,\infty[$ whose Euler characteristic is exactly $-n^2\chi(X)$.

Remark 5.6.2 The example of local systems is one of those cases where there is a canonical way to *derive* the usual moduli stack (see the discussion in Section 6). In fact, in this case we have $\mathcal{HOM}(\underline{S(X)}, \mathrm{Loc_n(pt)}) \simeq \mathrm{Loc}_n(X)$, for any CW-complex X, where $\mathcal{HOM}$ denotes the (underived) Hom-stack between (underived) stacks; therefore the natural thing to do is to first view the usual absolute stack $\mathrm{Loc}_n(pt)$ as a derived stack via the inclusion i and then derive the Hom-stack from $\underline{S(X)}$ to $i\mathrm{Loc}_n$. This authomatically gives an extension of $\mathrm{Loc}_n(X)$ i.e. a canonical derivation of it.

It is important to notice that the D-stack $\mathbb{R}\underline{Loc}_n(X)$ might be non-trivial even if X is simply connected. Indeed, the tangent at the unit local system is always the complex $C^*(X, \mathbb{C})[1]$. This shows that $\mathbb{R}\underline{Loc}_n(X)$ contains interesting information concerning the higher homotopy type of X. As noticed in the Introduction of [K-P-S], this is one of the reasons why the D-stack $\mathbb{R}\underline{Loc}_n(X)$ might be an interesting object in order to develop a version of non-abelian Hodge theory. We will therefore ask the same question as in [K-P-S].

Question 5.6.3 *Let X be a smooth projective complex variety and X^{top} its underlying topological space. Can one extend the non-abelian Hodge structure defined on the moduli space of local systems in [S3], to some kind of Hodge structure on the whole $\mathbb{R}\underline{Loc}_n(X)$?*

This question is of course somewhat imprecise, and it is not clear that the object $\mathbb{R}\underline{Loc}_n(X)$ itself could really support an interesting Hodge structure. However, we understand the previous question in a much broader sense, as for example it includes the question of defining derived versions of the moduli spaces of flat and Higgs bundles, and to study their relations from a non-abelian Hodge theoretic point of view, as done in [S3] for example.

5.6.2 Vector bundles on a projective variety

We now turn to the example of the derived stack of vector bundles, which is very close to the previous one. Let X be a fixed smooth projective variety.

If A is a cdga, we consider the space X (with the Zariski topology) together with its presheaf of cdga $\mathcal{O}_X \otimes A$. It makes sense to consider also presheaves of dg-$\mathcal{O}_X \otimes A$-modules on X and morphisms between them. We define a notion of equivalences between such presheaves, by saying the $f : \mathcal{M} \longrightarrow \mathcal{N}$ is an equivalence if it induces a quasi-isomorphism at each stalks. Using this notion of equivalences we can talk about equivalent dg-$\mathcal{O}_X \otimes A$-modules (i.e. objects which become isomorphic in the localization of the category with respect to quasi-isomorphisms).

We say that a presheaf of dg-$\mathcal{O}_X \otimes A$-module $\mathcal{M}$ on X is a *vector bundle of rank n*, if locally on $X_{zar} \times A_{\text{ét}}$ it is equivalent to $(\mathcal{O}_X \otimes A)^n$ (see the previous Subsection for details on this definition). We consider the category $wVect_n(X, A)$, of dg-$\mathcal{O}_X \otimes A$-modules which are vector bundles of rank n and flat (i.e. for each open U in X, the $\mathcal{O}_X(U) \otimes A$-module $\mathcal{M}(U)$ is cofibrant), and equivalences between them. By the standard strictification procedure we obtain a presheaf of categories

$$\begin{array}{ccc} CDGA & \longrightarrow & Cat \\ A & \mapsto & wVect_n(X, A) \\ (A \to B) & \mapsto & (\mathcal{M} \mapsto \mathcal{M} \otimes_A B). \end{array}$$

We then deduce a simplicial presheaf by appying the nerve construction

$$\begin{array}{rccc} \mathbb{R}\underline{Vect}_n(X) : & CDGA & \longrightarrow & Cat \\ & A & \mapsto & |wVect_n(X, A)|. \end{array}$$

This gives an object $\mathbb{R}\underline{Vect}_n(X) \in D - Aff^{\sim}$ that we call the derived moduli stack of rank n vector bundles on X.

We state the following result as a conjecture, as we do not have checked all details. However, we are very optimistic about it, as we think that a proof will probably consist of reinterpreting the constructions of [Ci-Ka1] in our context.

Conjecture 5.6.4 *1. The D-pre-stack $\mathbb{R}\underline{Vect}_n(X)$ is a strongly geometric, fp-smooth D-stack.*

2. There exists a natural isomorphism in $\mathrm{Ho}(D-Aff^{\sim})$

$$\mathbb{R}\underline{Vect}_n(X) \simeq \mathbb{R}\mathcal{HOM}(X, iBGl_n).$$

3. One has an equivalence

$$h^0\mathbb{R}\underline{Vect}_n(X) \simeq \underline{Vect}_n(X)$$

between the truncation of the D-stack $h^0\mathbb{R}\underline{Vect}_n(X)$ and the (Artin) stack of rank n vector bundles on X.

4. The tangent D-stack of $\mathbb{R}\underline{Vect}_n(X)$ at a vector bundle E on X, is the complex

$$C^*(X_{Zar}, \underline{End}(E))[1].$$

The same remark as in the case of the derived stack of local systems holds. Indeed, the usual Artin stack of vector bundles on X is given by $\mathbb{R}\mathcal{HOM}(X, BGl_n)$, and our D-stack of vector bundles on X is $\mathbb{R}\mathcal{HOM}(iX, iBGl_n)$.

5.6.3 Algebras and A_∞-categorical structures

In this last Subsection we present the derived moduli stack of associative algebra structures and A_∞-categorical structures. These are global versions of the formal moduli spaces studied in [Ko2, Ko-So].

Associative algebra structures. We are going to construct a D-stack $\mathbb{R}Ass$, classifying associative dg-algebra structures.

Let A be any cgda, and let us consider the category of (unbounded) associative differential graded A-algebras $A-Ass$ (i.e. $A-Ass$ is the category of monoids in the symmetric monoidal category $A-Mod$, of (unbounded) dg-A-modules)[7]. This category is a model category for which the weak equivalences are the quasi-isomorphisms and fibrations are epimorphisms. We restrict ourselves to the category of cofibrant objects $A-Ass^c$, and consider the sub-category $wA-Ass^c$ consisting of equivalences only. If $A \longrightarrow A'$ is any morphism of cdga's, then we have pull-back functors

$$wA - Ass^c \xrightarrow{-\otimes_A A'} wA' - Ass^c\ .$$

This defines a (lax) functor on the category of cdga's that we immediately strictify by the standard procedure. We will therefore assume that the above constructions are strictly functorial in A. By passing to the corresponding nerves we get a presheaf of simplicial sets

$$\begin{array}{rccc} \mathbb{R}Ass : & CDGA & \longrightarrow & SSet \\ & A & \mapsto & |wA - Ass^c|. \end{array}$$

This gives a well defined object $\mathbb{R}Ass$ in $D-Aff^{\sim}$.

We define a sub-simplicial presheaf $\mathbb{R}Ass_n$ of $\mathbb{R}Ass$, consisting of associative dg-A-algebras B for which there exists an étale covering $A \longrightarrow A'$ such that the dg-A'-module $B\otimes^{\mathbb{L}}_A A'$ is equivalent to $(A')^n$.

[7]By definition our associative A-dga's are then all central over A since they are commutative monoids in $A-Mod$.

Theorem 5.6.5 *1. The D-pre-stack $\mathbb{R}Ass_n$ is a D-stack.*

2. *The D-stack $\mathbb{R}Ass_n$ is strongly geometric. Furthermore, $h^0\mathbb{R}Ass_n$ is naturally equivalent to the (usual) Artin stack of associative algebra structures on $\mathbb{C}^n$.*
3. *For any global point $V : * \longrightarrow \mathbb{R}Ass_n$, corresponding to an associative $\mathbb{C}$-algebra V, the tangent D-stack of $\mathbb{R}Ass_n$ at V is the complex $\mathbb{R}Der(V,V)[1]$ of (shifted) derived derivations from V to V.*

¿From (3) we see that the geometric D-stack $\mathbb{R}Ass_n$ is not fp-smooth. Indeed, Quillen gives in [Qui1, Ex. 11.8] an example of a point in $\mathbb{R}Ass_n$ at which the dimension in the sense of Definition 5.5.10 is not defined.

The previous theorem can also be extended in the following way. Let V be a fixed cohomologically bounded and finite dimensional complex of $\mathbb{C}$-vector spaces. We define $\mathbb{R}Ass_V$ to be the sub-simplicial presheaf of $\mathbb{R}Ass$ consisting of associative dg-A-algebras B for which there exists an étale covering $A \longrightarrow A'$ such that the dg-A'-module $B \otimes^{\mathbb{L}}_A A'$ is equivalent to $A' \otimes V$.

On can that $\mathbb{R}Ass_V$ is again a D-stack, but it is not in general strongly geometric in the sense of Definition 5.5.1. However, one can show that it admits a smooth atlas and therefore is geometric in the sense of Definition 5.5.11. The tangent D-stack of $\mathbb{R}Ass_V$ at a point is given by the same formula as before.

The construction of $\mathbb{R}Ass_V$ can also be extended to classify algebra structures over an operad on the complex V. One checks that one also get geometric D-stacks this way. These are the geometric counterparts of the (discrete) moduli spaces described by C. Rezk in [Re].

A_∞-Categorical structures[8]**.** Let A by any cdga. Recall that a *dg-A-category* C consists of the following data

1. A set of objects $Ob(C)$.
2. For each pair of object (x,y) in $Ob(C)$, a (unbounded) dg-A-module $C_{x,y}$.
3. For each triplet of object (x,y,z) in $Ob(C)$, a composition morphism $C_{x,y} \otimes_A C_{y,z} \longrightarrow C_{x,z}$ which satisfies obvious associativity and unital conditions.

There is an obvious notion of *morphism* between dg-A-categories. There is also a notion of *equivalences* of dg-A-categories: they are morphisms $f : C \longrightarrow C'$ satisfying the following two conditions

1. For any pair of objects (x,y) of C, the induced morphism $f_{x,y} : C_{x,y} \longrightarrow C'_{x,y}$ is a quasi-isomorphism of dg-A-modules.
2. Let $H^0(C)$ (resp. $H^0(C')$) be the categories having repectively the same set of objects as C (resp. as C'), and $H^0(C_{x,y})$ (resp. $H^0(C'_{x,y})$) as set of morphisms from x to y. Then, the induced morphism
$$H^0(f) : H^0(C) \longrightarrow H^0(C')$$
is an equivalence of categories (in the usual sense).

Using these definitions, one has for any cdga A, a category $A - Cat$ of dg-A-categories, with a sub-category of equivalences $wA - Cat$. Furthermore, if $A \longrightarrow A'$ is a morphism of cdga, one has a

[8] We are working here with the stronger notion of dg-category (or *strict A_∞-categories*), and of course one could also use A_∞-categories instead. However, as the homotopy theories of dg-categories and A_∞-categories are equivalent, the D-stacks obtained would be the same.

pull-back functor $A-Cat \longrightarrow A'-Cat$, obtained by tensoring the dg-A-modules $C_{x,y}$ with A'. With a bit of care (e.g. by restricting to *cofibrant* dg-A-categories), one gets a simplicial presheaf

$$\begin{array}{rccc} \mathbb{R}Cat: & CDGA & \longrightarrow & SSet \\ & A & \mapsto & |wA-Cat|, \end{array}$$

that is an object in $D-Aff^{\sim}$.

We now consider the sub-simplicial presheaf $\mathbb{R}Cat_f$ of $\mathbb{R}Cat$, consisting of all dg-A-categories C such that all the dg-A-modules $C_{x,y}$ are finitely presented (i.e. are retracts of finite cell dg-A-modules [Kr-Ma, §III.1], or equivalently are strongly dualizable or equivalently perfect).

In order to state the following result, we mention that iterating Definition 5.5.11 leads to a notion of n-geometric D-stacks. We will use the notion of a 2-geometric D-stacks, which is defined to be a D-stack having a 1-representable diagonal (i.e. pull-backs of the morphism $F \longrightarrow F \times F$ along all $\mathbb{R}Spec\, A \longrightarrow F \times F$ are 1-geometric D-stacks), and which admits a smooth 1-geometric atlas (i.e. a smooth morphism from a disjoint union of 1-geometric stacks, which is a covering). The reader will find the full definition of higher geometric D-stacks in the forthcoming paper [HAG-II], or might himself reconstruct the definition from [S2].

Theorem 5.6.6 *Let $\widetilde{\mathbb{R}Cat_f}$ be the associated D-stack to the D-pre-stack $\mathbb{R}Cat_f$.*

1. *The D-stack $\widetilde{\mathbb{R}Cat_f}$ is 2-geometric.*
2. *For any global point $C : * \longrightarrow \mathbb{R}Cat_f$, corresponding to a dg-category C, the tangent D-stack of $\widetilde{\mathbb{R}Cat_f}$ at C is the whole (shifted) Hochschild cohomology complex $C^*(C,C)[2]$* (see e.g. [Ko-So, 2.1] or [So, 2]).

Remark 5.6.7 For a cgda A, points in $\widetilde{\mathbb{R}Cat_f}(A)$ can be described as certain *twisted forms* of dg-A-categories on the étale site of A.

Note that the D-stack $\widetilde{\mathbb{R}Cat_f}$ cannot be 1-geometric, as its truncation $h^0\widetilde{\mathbb{R}Cat_f}$ has a component corresponding to the 2-geometric stack of linear categories. As a 1-geometric stack is always 1-truncated (contrary to the derived situation), this shows that $\widetilde{\mathbb{R}Cat_f}$ must be at least 2-geometric.

We let $\mathbb{R}Ass_f$ be the sub-simplicial presheaf of $\mathbb{R}Ass$ defined before, consisting of associative A-dga's which are finitely presented as dg-A-modules. Then, there exists a natural morphism

$$\mathbb{R}Ass_f \longrightarrow \widetilde{\mathbb{R}Cat_f},$$

that sends an associative dga to the dg-category, with one object, it defines. This morphism is actually a *gerbe* in the following sense. If $B : \mathbb{R}Spec\, A \longrightarrow \mathbb{R}Ass_f$ corresponds to associative A-dga, then the homotopy fiber F of the previous morphism is locally equivalent to the D-stack over $\mathbb{R}Spec\, A$ sending an cdga $A \to A'$ to the simplicial set $K((B\otimes_A A')^*, 1)$, where $(B\otimes_A A')^*$ is the loop space of invertible elements in $B \otimes_A A'$ (i.e. the mapping space $\mathrm{Map}_{A'-alg}(A'[T,T^{-1}], B\otimes_A A')$). In particular, one deduces that the morphism $\mathbb{R}Ass_f \longrightarrow \widetilde{\mathbb{R}Cat_f}$ is a smooth morphism of 2-geometric stacks (a smooth morphisms between 2-geometric stacks is defined in an analogous way as for 1-geometric stacks). This smooth morphism induces in particular an exact triangle between the tangent D-stacks

$$\mathbb{R}TF_B \longrightarrow \mathbb{R}T(\mathbb{R}Ass_f)_B \longrightarrow \mathbb{R}T(\widetilde{\mathbb{R}Cat_f})_B \xrightarrow{+1}$$

which can also be written

$$B[1] \longrightarrow \mathbb{R}Der_A(B,B)[1] \longrightarrow C_A^+(B,B)[2] \xrightarrow{+1}$$

which is our way of understanding the triangle appearing in [Ko2, p. 59] (at least for $d = 1$).

The fact that the tangent D-stack of $\widetilde{\mathbb{R}Cat_f}$ at a dg-category with only one object is the whole (shifted) Hochschild complex $C^*(A, A)[2]$, where A is the dg-algebra of endomorphism of the unique object, is also our way to understand the following sentence from [Ko-So, p. 266]:

In some sense, the full Hochschild complex controls deformations of the A_∞-category with one object, such that its endomorphism space is equal to A.

We see that the previous results and descriptions produce global versions of the formal moduli spaces of A_∞-categories studied for example in [Ko-So, So]. This also shows that there are interesting 2-geometric stacks, and probably interesting n-geometric stacks (as, for example, the n-geometric stack of $(n-1)$-*dg-categories*, whatever these are) as suggested by a higher analog of the exact triangle above (see [Ko2, 2.7 Claim 2]).

5.7 Final comments on deriving moduli functors

In this Subsection, keeping in mind the examples presented before, we would like to discuss, from a more general point of view, the problem of derivation of moduli functors, with the aim of at least making explicit some general features shared by the examples.

Suppose $M : (Aff)^{\mathrm{op}} = (\mathbb{C}-alg) \longrightarrow Set$ is a functor arising from some geometric moduli problem e.g., the problem of classifying isomorphism classes of families of (pointed) curves of a given genus. Very often, the *moduli functor* M is not representable and only admits a coarse moduli space. As its name says, when passing to a coarse moduli space some information is lost. The theory of *stacks in groupoids* was originally invented to correct this annoyance, by looking at natural *extensions* of M, i.e. to functors $\mathcal{M}_1$, from algebras to groupoids, such that the following diagram commutes

$$\begin{array}{ccc} Aff^{\mathrm{op}} & \xrightarrow{M} & Set \\ & \searrow^{\mathcal{M}_1} & \uparrow{\scriptstyle \pi_0} \\ & & Grpd, \end{array}$$

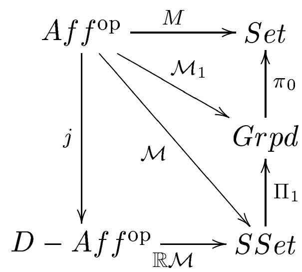
Here the vertical arrow assigns to a given groupoid its set of isomorphisms classes of objects. Of course, the point of the theory of stacks in groupoids is precisely to develop a *geometry* on this kind of functors.

More generally, other natural *higher* moduli problems are not representable even when considered as stacks in groupoids, e.g. the 2-stack perfect complexes of length 1, the 2-stack of linear categories ...; the theory of *higher stacks* precisely says that one should consider M extended as follows

$$\begin{array}{ccc} Aff^{\mathrm{op}} & \xrightarrow{M} & Set \\ & \searrow^{\mathcal{M}_1} & \uparrow{\scriptstyle \pi_0} \\ & & Grpd \\ & \searrow^{\mathcal{M}} & \uparrow{\scriptstyle \Pi_1} \\ & & SSet, \end{array}$$

where the functor Π_1 maps a simplicial sets to its fundamental groupoid. The notion of geometric n-stacks of [S2] can then be used in order to set up a *geometry* over these kind of objects, in pretty much the same way one is doing geometry over stacks in groupoids.

The idea of derived algebraic geometry is to seek for derived extensions of M, $\mathcal{M}_1$ and $\mathcal{M}$ i.e. to extend not (only) the target category of this functors but more crucially the source category in a "derived" direction. More precisely, we define a *derived extension* of a functor $\mathcal{M} : Aff^{op} \longrightarrow SSet$, as above, to be a functor $\mathbb{R}\mathcal{M} : (D - Aff)^{\mathrm{op}} \longrightarrow SSets$ making the following diagram commute



where j denotes the natural inclusion (a $\mathbb{C}$-algebra viewed as a cdga concentrated in degree zero). The above diagram shows that, for any derived extension $\mathbb{R}\mathcal{M}$, we have

$$\pi_0\mathbb{R}\mathcal{M}(j(\mathrm{Spec}R)) \simeq M(\mathrm{Spec}R)$$

and moreover

$$\Pi_1\mathbb{R}\mathcal{M}(j(\mathrm{Spec}R)) \simeq \mathcal{M}_1(\mathrm{Spec}R)$$

for any commutative $\mathbb{C}$-algebra R. In other words, the 0-truncation of $\mathbb{R}\mathcal{M}$ gives back M when restricted to the image of j, while its 1-truncation gives back $\mathcal{M}_1$.

What about the existence or uniqueness of a derived extension $\mathbb{R}\mathcal{M}$? First of all, extensions always exists, as one can use the *trivial* one given by the functor i of §3.2. But of course, this extension is far from being unique and usually does not give the *expected answer*. However, there is no canonical choice for an extension which could be *nicer* than others. This tells us that the choice of the extended moduli functor $\mathbb{R}\mathcal{M}$ highly depends on the geometrical meaning of the original moduli functor M, $\mathcal{M}_1$ of $\mathcal{M}$. We would like to give here a clear example to show this.

Let S^2 be the 2-dimensional sphere, and let us consider $\mathcal{M}_1 := \underline{Loc}_n(S^2)$, the moduli stack of rank n local systems on S^2. We clearly have $\mathcal{M}_1 \simeq BGL_n$. If one thinks of $\mathcal{M}_1$ simply as BGl_n, and forget about the fact that it is the moduli stack of local systems on S^2, then a reasonable extension of $\mathcal{M}_1$ is simply $iBGl_n \simeq \mathbb{R}BGl_n$ as described in §3.4. However, if one remembers that $\mathcal{M}_1$ is $\underline{Loc}_n(S^2)$, then the *correct* (or at least *expected*) extension is $\mathbb{R}\underline{Loc}_n(S^2)$ presented in Theorem 5.6.1. Definitely, these two extensions are very different. This example shows that the *expected* extension $\mathbb{R}\mathcal{M}$ depends very much on the way we *think* of the original moduli problem $\mathcal{M}$. In a way we are more *deriving our interpretation* of the moduli functor rather than the moduli functor itself. Another example of the existence of multiple choices can be found in [Ci-Ka2], in which the derived Hilbert dg-scheme is not the same as the derived $Quot_{\mathcal{O}}$ dg-scheme.

Nevertheless, the derived extension of a moduli functor that typically occurs in algebraic geometry, is expected to satisfy certain properties and this gives some serious hints in order to guess the *correct answer.*

First of all, in general, one knows a priori what is the *expected derived tangent stack* $T\mathbb{R}\mathcal{M}$ (or, at least, the disembodied derived tangent complexes at the points, the $(\Omega^1_{\mathbb{R}\mathcal{M},x})^*$'s in the notations of §4.3); namely, this is true in the case where $\mathcal{M}$ classifies vector bundles over a scheme, local systems over a topological space, families curves or higher dimensional algebraic varieties, stable maps from a fixed scheme and so on. For some examples of the expected derived tangent spaces we refer again to [Ci-Ka1, Ci-Ka2]. To put it slightly differently, it is always the case that one looks for a derived extension by simply requiring it to have the expected derived tangent stack. This is essentially due to the fact that the correct derived deformation theory of the moduli problem has already been guessed, and the corresponding, already established, formal theoy is based on this guess (see [Hin2, Ko-So, So], to quote a few).

Even if this does not say exactly how to construct a derived extension, it certainly puts some constraints on the possible choices. To go a bit further, one may notice that all the usual moduli functor occurring in algebraic geometry classify *families of geometric objects* over varying base schemes. To produce a derived extension $\mathbb{R}\mathcal{M}$, the main principle is then the following

Main principle: *Let $\mathcal{M}$ be a moduli stack classifying certain kind of families of geometric objects over varying commutative algebras A. In order to guess what the extended moduli stack $\mathbb{R}\mathcal{M}$ should be, guess first what is a family of geometric objects of the same type parametrized by a commutative dga A.*

In the case, for example, where the classical notion of a family is defined through the existence of a map with some properties (like for example in the case of the stack of curves), the derived analog is more or less clear: one should say that a derived family over a cdga A is just a map of simplicial presheaves $F \longrightarrow \mathbb{R}Spec\, A$, having the same properties in the derived sense (e.g., as we extended the notion of étale morphism of schemes to cdga's, see §2.2, the same can be done with the notions of smooth, flat ... morphisms of schemes). Then, a natural candidate for a derived notion of family of geometric objects, is given by *any* derived analog of a family such that *when restricted along* $\mathrm{Spec}(H^0(A)) \longrightarrow \mathrm{Spec}A$ it becomes *equivalent* to an *object coming from* $\mathcal{M}(\mathrm{Spec}(H^0(A)))$. This condition, required in order to really get a derived extension, essentially says that the derived version of a family of geometric objects should reduce to a non-derived family of geometric objects in the non-derived or scheme-like direction, i.e. along $\mathrm{Spec}(H^0(A)) \longrightarrow \mathrm{Spec}A$. A typical example of this case is the one of G-torsors given in §3.4. Another example would be that of the moduli stack of surfaces. One could say for example that a smooth projective family of surfaces over a cdga A, is a strongly smooth morphism of strongly geometric D-stacks $F \longrightarrow \mathbb{R}Spec\, A$, such that for any geometric point $x : Spec\,\mathbb{C} \longrightarrow \mathbb{R}Spec\, A$, the pull-back $F \times^h_{\mathbb{R}Spec\, A} Spec\,\mathbb{C}$ is equivalent to a smooth projective surface.

Though this gives perhaps only a vague recipee of a possible construction of derived extensions of some of the moduli functors occurring in algebraic geometry, we thought it was worthwhile presenting it, if not certainly to solve the problem at least to pose it in a general perspective.

References

[SGA1] M. Artin, A. Grothendieck, *Revêtements étales et groupe fondamental*, Lecture notes in Math. **224**, Springer-Verlag 1971.

[SGA4-I] M. Artin, A. Grothendieck, J. L. Verdier, *Théorie des topos et cohomologie étale des schémas- Tome 1*, Lecture Notes in Math **269**, Springer Verlag, Berlin, 1972.

[SGA4-II] M. Artin, A. Grothendieck, J. L. Verdier, *Théorie des topos et cohomologie étale des schémas- Tome 2*, Lecture Notes in Math **270**, Springer Verlag, Berlin, 1972.

[Be] K. Behrend, *A 2-categorical approach to dg-schemes*, in preparation, see also http://www.msri.org/publications/ln/msri/2002/intersect/behrend/1/index.html.

[Be-Fa] K. Behrend, B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), No. 1, 45-88.

[Bl] B. Blander, *Local projective model structure on simpicial presheaves*, K-theory **24** (2001) No. 3, 283-301.

[Ci-Ka1] I. Ciocan-Fontanine, M. Kapranov, *Derived Quot schemes*, Ann. Sci. Ecole Norm. Sup. (4) **34** (2001), 403-440.

[Ci-Ka2] I. Ciocan-Fontanine, M. Kapranov, *Derived Hilbert Schemes*, preprint math.AG/0005155.

[De] P. Deligne, *Catégories Tannakiennes*, in *Grothendieck Festschrift Vol. II*, Progress in Math. **87**, Birkhauser, Boston 1990.

[DHI] D. Dugger, S. Hollander, D. Isaksen, *Hypercovers and simplicial presheaves*, preprint math.AT/0205027.

[D-K1] W. Dwyer, D. Kan, *Simplicial localisation of categories*, J. Pure and Appl. Algebra **17** (1980), 267-284.

[D-K2] W. Dwyer, D. Kan, *Calculating simplicial localizations*, J. Pure and Appl. Algebra **18** (1980), $17-35$.

[D-K3] W. Dwyer, D. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), $427-440$.

[D-K4] W. Dwyer, D. Kan, *Equivalences between homotopy theories of diagrams*, in *Algebraic topology and algebraic K-theory*, *Annals of Math. Studies* **113**, Princeton University Press, Princeton, 1987, 180-205.

[D-K5] W. Dwyer, D. Kan, *Homotopy commutative diagrams and their realizations*, J. Pure Appl. Algebra **57** (1989) No. 1, 5-24.

[DHK] W. Dwyer, P. Hirschhorn, D. Kan, *Model categories and more general abstract homotopy theory*, Book in preparation, available at http://www-math.mit.edu/~psh.

[D-S] W. Dwyer, J. Spalinski, *Homotopy theories and model categories*, in Handbook of algebraic topology, 73–126, North-Holland, Amsterdam, 1995.

[EKMM] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, *RI*, 1997.

[Ga-Zi] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Math. und ihrer Grenzgebiete **35**, Springer-Verlag, New-York 1967

[G-J] P. Goerss, J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, Vol. **174**, Birkhauser Verlag 1999.

[Gr1] A.Grothendieck, *Catégories cofibrées additives et complexe cotangent relatif*, Lecture Note in Mathematics 79, Springer-Verlag, Berlin, 1968.

[Gr2] A. Grothendieck, *Les Derivateurs*, Édité par M. Künzer, J. Malgoire, G. Maltsiniotis , available at http://www.math.jussieu.fr/ maltsin/groth/Derivateurs.html

[Ha] M. Hakim, *Topos annelés et schémas relatifs*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 64. Springer-Verlag Berlin-New York, 1972.

[He] A. Heller, *Homotopy theories*, Memoirs AMS, **71**, N. 383 (1988), 1–77.

[Hin1] V. Hinich, *Homological algebra of homotopical algebras*, Comm. in Algebra **25** (1997), 3291-3323.

[Hin2] V. Hinich, *Formal stacks as dg-coalgebras*, J. Pure Appl. Algebra **162** (2001), No. 2-3, 209-250.

[Hi] P. S. Hirschhorn, *Model Categories and Their Localizations*, book to appear (references in the text are to the August 2002 preprint version available at http://www-math.mit.edu/~psh).

[H-S] A. Hirschowitz, C. Simpson, *Descente pour les n-champs*, preprint math.AG/9807049.

[Hol] S. Hollander, *A homotopy theory for stacks*, preprint math.AT/0110247.

[Ho] M. Hovey, *Model categories*, Mathematical surveys and monographs, Vol. **63**, Amer. Math. Soc. Providence 1998.

[Ho-Sh-Sm] M. Hovey, B.E. Shipley, J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149-208.

[Ja1] J. F. Jardine, *Simplicial presheaves*, J. Pure and Appl. Algebra **47** (1987), 35-87.

[Ja2] J. F. Jardine, *Stacks and the homotopy theory of simplicial sheaves*, in *Equivariant stable homotopy theory and related areas* (Stanford, CA, 2000). Homology Homotopy Appl. **3** (2001), No. 2, 361-384.

[Jo1] A. Joyal, Letter to Grothendieck.

[Jo-Ti] A. Joyal, M. Tierney, *Strong stacks and classifying spaces*, in *Category theory (Como, 1990)*, Lecture Notes in Mathematics **1488**, Springer-Verlga New York, 1991, 213-236.

[Ka1] M. Kapranov, *dg-Schemes in algebraic geometry*, talk at MSRI, March 2002, notes available at http://www.msri.org/publications/ln/msri/2002/intersect/kapranov/1/index.html.

[Ka2] M. Kapranov, *Injective resolutions of BG and derived moduli spaces of local systems*, J. Pure Appl. Algebra **155** (2001), No. 2-3, 167-179.

[Ka-Vo] M. Kapranov, V. Voevodsky, 2-*Categories and Zamolodchikov tetrahedra equations*, in *Algebraic groups and their generalizations: Quantum and infinte dimensional methods*, University Park PA, 1991, 177-269, Proc. Sympos. Pure Math. **56** part 2, AMS, Providence, RI, 1994.

[K-P-S] L. Katzarkov, T. Pantev, C. Simspon, *Non-abelian mixed Hodge structures*, preprint math.AG/0006213.

[Ke] G. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, **64**, Cambridge Universtity Press, Cambridge-New York, 1982.

[Ko1] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), 335-368, Progr. Math. **129**, Birkhauser Boston, MA, 1995.

[Ko2] M. Kontsevich, *Operads and motives in deformation quantization*, Moshé Flato (1937-1998), Lett. Math. Phys. **48**, (1999), No. 1, 35-72.

[Ko-So] M. Kontsevich, Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. 1 (Dijon), 255-307, Math. Phys. Stud. **21**, Kluwer Acad. Publ, Dordrecht, 2000.

[Kr-Ma] I. Kriz, J. P. May, *Operads, algebras, modules and motives*, Astérisque **233**, 1995.

[La-Mo] G. Laumon and L. Moret-Bailly, *Champs algébriques*, A series of Modern Surveys in Mathematics vol. **39**, Springer-Verlag 2000.

[M-M] S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic*, Springer, New York 1992.

[Mal] G. Maltsiniotis *Introduction á la théorie des dérivateurs* (d'aprés Grothendieck), Preprint (2001), available at http://www.math.jussieu.fr/ maltsin/index.html

[Man] M. Manetti, *Extended deformation functors*, Int. Math. Res. Not. **14** (2002), 719-756.

[MCM] R. Mc Carthy, V. Minasian, *Smooth S-algebras*, preprint 2002.

[Mil] J. S. Milne, *Étale cohomology*, Princeton University Press, 1980.

[Qui1] D. Quillen, *On the (co-)homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol XVII, New York, 1964), 65-87. Amer. Math. Soc., Providence, P.I.

[Qui2] D. Quillen, *Homotopical algebra*, LNM 43, Springer, 1967.

[Re] C. Rezk, *Spaces of algebra structures and cohomology of operads*, Thesis 1996, available at http://www.math.uiuc.edu/ rezk.

[Sch] H. Schubert, *Categories*, Springer Verlag, Berlin, 1970.

[S1] C. Simpson, *Homotopy over the complex numbers and generalized cohomology theory*, in *Moduli of vector bundles (Taniguchi Symposium, December 1994)*, M. Maruyama ed., Dekker Publ. (1996), 229-263.

[S2] C. Simpson, *Algebraic (geometric) n-stacks*, preprint math.AG/9609014.

[S3] C. Simpson, *The Hodge filtration on non-abelian cohomology*, preprint math.AG/9604005.

[So] Y. Soibelman, *Mirror symmetric and deformation quantization*, preprint hep-th/0202128.

[TT] R. W. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, pp. 247-435 in *The Grothendieck Festschrift*, Vol. III, Birkhäuser, 1990.

[To] B. Toen, *Dualité de Tannaka supérieure I: Structures monoïdales*, Preprint MPI für Mathematik (57), Bonn, 2000, available at http://www.mpim-bonn.mpg.de.

[To-Ve 1] B. Toen, G. Vezzosi, *Algebraic geometry over model categories. A general approach to Derived Algebraic Geometry*, preprint available at math.AG/0110109.

[HAG-I] B. Toën, G. Vezzosi, *Homotopy Algebraic Geometry I: Topos theory*, preprint math.AG/0207028.

[HAGDAG] B. Toën, G. Vezzosi, *From HAG to DAG: derived moduli spaces*, preprint math.AG/0210407.

[HAG-II] B. Toën, G. Vezzosi, *Homotopy Algebraic Geometry II: Geometric stacks*, in preparation.

[Vo] R. M. Vogt, *Introduction to algebra over "brave new rings"*, The 18th winter school "Geometry and physics" (Srni, 1998), Rend. Circ. Mat. Palermo (2) Suppl. **59** (1999), 49-82.