WHAT IS...

the Langlands Program?

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The Langlands Program envisions deep links between arithmetic and analysis, and uses constructions in arithmetic to predict maps between spaces of functions on different groups. The conjectures of the Langlands Program have shaped research in number theory, representation theory, and other areas for many years, but they are very deep, and much still remains to be done.

For arithmetic, if K is a finite Galois extension of the rationals we have the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of all ring automorphisms of K fixing $\mathbb{Q}$. Frobenius suggested studying groups by embedding them into groups of matrices, so fix a homomorphism $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to GL(V)$ where V is a finite dimensional complex vector space. For almost all primes p one can define a certain conjugacy class Frob_p in the Galois group. Artin introduced a function in a complex variable s built out of the values of the characteristic polynomials for $\rho(\mathrm{Frob}_p)$: $L(s,\rho) = \prod_p' \det(\mathrm{I}_V - \rho(\mathrm{Frob}_p)p^{-s})^{-1}$. Here the product is an Euler product, that is a product over the primes p; it converges for $\Re(s) > 1$. Also, there is a specific adjustment at a finite number of primes (indicated by the apostrophe in the product above) which we suppress. For example, if V is one dimensional and ρ is trivial, then $L(s,\rho)$ is exactly the Riemann zeta function. Artin made the deep conjecture that these "Artin L-functions" (which Brauer showed have meromorphic continuation to $s \in \mathbb{C}$) are entire if ρ is irreducible and nontrivial.

On the analysis side, let G be a (nice) algebraic group such as GL_n. Then one can study the space $L^2(\Gamma\backslash G(\mathbb{R}))$ consisting of complex-valued functions on $G(\mathbb{R})$ that are invariant under a large discrete subgroup Γ (such as a finite index subgroup of $GL_n(\mathbb{Z})$), transform under the center by a character, and that are square-integrable with respect to a natural group-invariant measure. The group $G(\mathbb{R})$ acts by the right regular representation, and it turns out that to each invariant subspace π (with some conditions) one can once again attach an Euler product $L(s,\pi)$ that converges for $\Re(s)$ sufficiently large, called the standard L-function of π. The space of functions π is called an automorphic representation. (One may also consider certain other functions on $\Gamma\backslash G(\mathbb{R})$ and certain quotient spaces.) The subject of automorphic forms studies the functions in π, and shows that for most π the "automorphic" L-series $L(s,\pi)$ is entire. A first example is $G = GL_1$, and there one recovers (from the adelic version) the theory of Dirichlet L-functions attached to a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, and used by Dirichlet to prove his primes in progressions theorem.

Langlands's first insight is that Artin's conjecture should be true because Galois representations are connected to automorphic representations. More precisely, he conjectures that each Artin L-function $L(s,\rho)$ should in fact be an automorphic L-function $L(s,\pi)$, with π on $GL_{\dim V}$. If $\dim V = 1$ this assertion states that the Artin L-function is in fact a Dirichlet L-function; this is Artin's famous reciprocity law. Incidentally, the converse to this conjecture is false—there are far more analytic objects than algebraic ones.

This is already remarkable. But Langlands suggests much more. There are natural (and easy) algebraic constructions on the arithmetic side. Langlands conjectures that there should be matching (but, it seems, not easy) constructions on the analytic side.

For example, suppose V and W are two finite dimensional vector spaces and $\sigma : GL(V) \to GL(W)$ is a group homomorphism. (Concretely, W could be the symmetric nth power $\mathrm{Sym}^n(V)$ for some fixed n and σ the natural map.) Then there is a map on the Galois side given by composition: $\rho \mapsto \sigma \circ \rho$. If each of these Galois representations corresponds to an automorphic representation,

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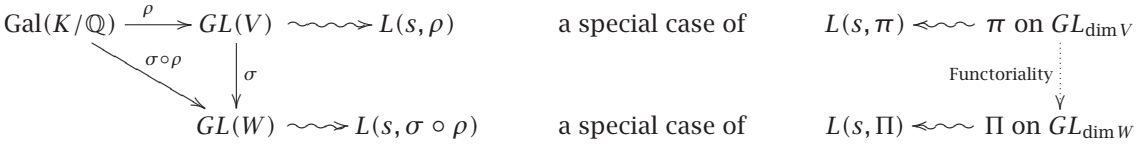

Figure 1. Natural maps for Galois representations lead to conjectured functoriality maps.

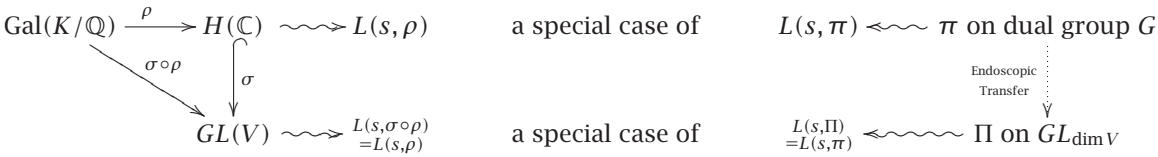

Figure 2. Endoscopic transfer: automorphic representations on different groups with the same L-function.

then there should be a map taking those spaces of functions attached to $GL_{\dim V}$ to certain spaces of functions attached to $GL_{\dim W}$. The Langlands Functoriality Conjecture asserts that there should be a full matching map on the analytic side, taking general automorphic representations π on quotients of $GL_{\dim V}(\mathbb{R})$ to automorphic representations Π on quotients of $GL_{\dim W}(\mathbb{R})$, such that the L-functions correspond. (More about this momentarily.) See Figure 1. There is no simple reason why the analysis on these two groups, each modulo a discrete subgroup, should be related; indeed, even if one can find a natural way to construct a function on $GL_{\dim W}(\mathbb{R})$ from one on $GL_{\dim V}(\mathbb{R})$, it is quite difficult to make one that is invariant under a large discrete subgroup of $GL_{\dim W}(\mathbb{R})$ and square-integrable on the quotient. The existence of this map in general would have important consequences including the generalized Ramanujan Conjecture. It is known in only a few cases.

The correspondence of L-functions is a key feature of the Langlands Functoriality Conjecture. For the conjectural map $\pi \mapsto \Pi$ described above, this means that if (for $\Re(s)$ sufficiently large)

$$L(s,\pi) = \prod_p{}' \det(\mathrm{I}_{\dim V} - A_p p^{-s})^{-1}$$

where A_p is an invertible diagonal matrix for all but finitely many primes p, then the new product

$$L(s,\pi,\sigma) := \prod_p{}' \det(\mathrm{I}_{\dim W} - \sigma(A_p) p^{-s})^{-1}$$

matches $L(s,\Pi)$ for almost all primes p. In fact, similarly to the Riemann zeta function, all automorphic L-functions should have analytic continuation to $\mathbb{C}$ and satisfy functional equations under $s \mapsto 1-s$. We know this property for $L(s,\pi)$ but not for $L(s,\pi,\sigma)$ in general (let alone that $L(s,\pi,\sigma)$ matches the standard L-function for an automorphic representation Π on $GL_{\dim W}$). The continuation of $L(s,\pi,\sigma)$ to $\mathbb{C}$ for a given symmetric power σ already carries important analytic information.

Just as Langlands predicts a map of spaces of functions that corresponds to the homomorphism σ, he also predicts maps of automorphic representations corresponding to other algebraic operations: a map that is a counterpart of the algebraic operation of tensor product, and maps corresponding to induction and restriction of Galois representations. These last maps require a tower of field extensions and base fields other than $\mathbb{Q}$, a topic that is best addressed by passing to the adeles. The automorphic induction and restriction maps for the general linear groups were constructed for cyclic extensions by Arthur and Clozel using the trace formula, following earlier work for GL_2 by Langlands and others.

So far we have only made use of general linear groups. Langlands also addresses how other algebraic groups enter the story. Suppose that the image of the Galois representation ρ is contained in a subgroup $H(\mathbb{C})$ of $GL(V)$, which is the complex points of a (nice) algebraic group H. Then Langlands conjectures that there should be an automorphic representation π on a group G, the dual group of H (the definition involves Lie theoretic data), whose L-function matches that of ρ. For example, suppose the image of ρ stabilizes a nondegenerate symplectic form on V, so after a choice of basis, ρ maps into the symplectic group $Sp_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ for some even number n. Then Langlands conjectures that ρ should correspond (in the sense of matching L-functions) to an automorphic representation π on the special orthogonal group SO_{n+1}.

But wait a moment! If we simply forget that ρ maps into a subgroup $H(\mathbb{C})$ of $GL(V)$, then ρ should also correspond to an automorphic representation Π of $GL_{\dim V}$. Langlands then suggests that every automorphic representation π of G should correspond to an automorphic representation Π of $GL_{\dim V}$, with $L(s,\pi) = L(s,\Pi)$. See Figure 2. To summarize, given an automorphic representation on a symplectic or orthogonal group, there should be a corresponding automorphic representation on a general linear group with the same L-function, corresponding to the natural inclusion on the dual group side. Such maps are called endoscopic liftings (they 'see inside' the general linear group). They were established for certain classes of automorphic representations by Cogdell, Kim,

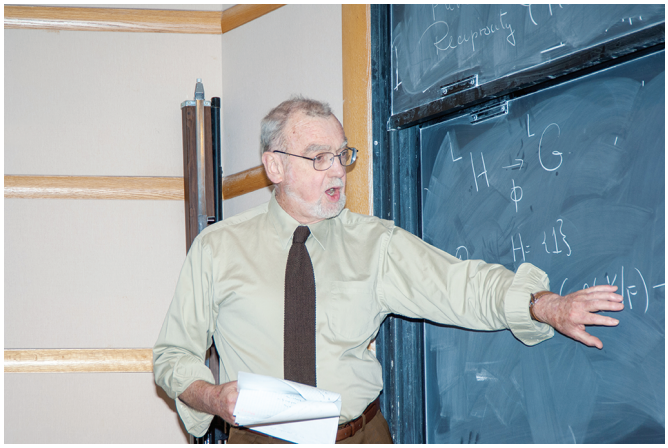

Figure 3. In the 1960s and 1970s Robert Langlands proposed a program relating arithmetic and certain spaces of functions on groups called automorphic representations.

Piatetski-Shapiro and Shahidi, and in full generality using the trace formula by Arthur.

More generally, whenever there is a homomorphism of dual groups the Langlands Functoriality Conjecture asserts that there should be a corresponding map of automorphic representations. That is, spaces of functions on the quotients of different real (or more generally, adelic) groups by discrete subgroups should be related. A further extension of functoriality replaces the dual group by the L-group, which involves the Galois group, so that the automorphicity of Artin L-functions is part of the same functoriality conjecture. Langlands is describing this in Figure 3.

The endoscopic lifting established by Arthur is a significant step in the Langlands Program. However, most of the program, including most cases of establishing functorial liftings of automorphic representations and most cases of the matching of Artin and automorphic L-functions, remains unproved. There is also a conjectured vast generalization of this matching, that every motivic L-function over a number field is an automorphic L-function (this includes Wiles's Theorem as one case), that is mostly unproved.

The Langlands Program also includes local questions, which allow one to say more about the L-functions at the finite number of primes p that were not discussed above (and where much progress has recently been announced by Scholze), and a version where a number field is replaced by the function field attached to a curve over a finite field, resolved for the general linear groups by L. Lafforgue. Another direction is the p-adic Langlands Program where one considers p-adic representations.

Langlands's vision connects arithmetic and analysis, and begins from Galois groups. We also know that Galois-like groups may be attached to coverings of Riemann surfaces. This substitution leads to the geometric Langlands Program, with connections to physics. The richness of Langlands's fundamental idea that simple natural maps for Galois representations are the shadows of maps for all automorphic representations continues to expand.

EDITOR'S NOTE. Robert Langlands in March was named winner of the 2018 Abel Prize for his program. More on Langlands is planned for the 2019 *Notices*. Meanwhile see the feature on James G. Arthur and his work on the Langlands Program in this issue.

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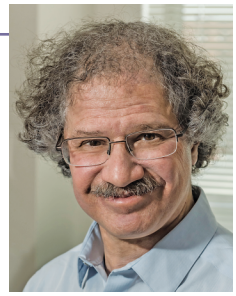
ABOUT THE AUTHOR

In addition to his work in number theory and representation theory and involvement in K–12 math education, **Solomon Friedberg** enjoys outdoor activities, the arts, and foreign travel.



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