NT

Lent '92 (Caltech)

Number Theory — Richard Taylor

lecture 1
13/1/92

The aim - to try & understand the absolute Galois gp of $\mathbb{Q}$ (or any number field)

The first step of this was done at the beginning of the century -

$$\underline{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{ab}} \cong \prod_p \mathbb{Z}_p^\times$$

↑ a quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, corresponding to the field $\mathbb{Q}^{ab}$.

This is class field theory.

In $\mathbb{Q}$ case, the Kronecker-Weber thm tells us that $\mathbb{Q}^{ab} = \mathbb{Q}(\text{roots of unity})$

The generalisation of this to number fields was one of the high points of class field thy earlier this century.

We really want to understand the whole of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Langlands' idea - look at n-dim^l rep's (cts..)

Langlands:

n-dim^l rep's of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ $\xleftrightarrow{\text{some correspondence}}$ certain rep's of $GL_n(\mathbb{A})$ ← adèles

$\mathbb{A}$ is a large topological ring : $\mathbb{A} = \left\{ \underset{\sim}{x} \in \prod_p \mathbb{Q}_p \times \mathbb{R} \;\middle|\; x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\}$

Note $\prod_p \mathbb{Z}_p \times \mathbb{R} \subseteq \mathbb{A}$. Topologise $\mathbb{A}$ by saying that $\prod_p \mathbb{Z}_p \times \mathbb{R}$ is an open subgp with the usual topology.

$GL_n(\mathbb{A})$ now inherits a topology (not the most naïve idea though)

Why is Langlands' idea a generalisation of class field theory?

Say $n=1$. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} \leftrightsquigarrow$ certain rep's of $GL_1(\mathbb{A}) = \mathbb{A}^\times$.

Fact:
$$\mathbb{Q}^\times \mathbb{R}^\times_{>0} \backslash \mathbb{A}^\times \cong \prod_p \mathbb{Z}_p^\times \quad ①$$

so class field theory is just the case $n=1$.

Richard wants to talk mostly about the case $n=2$.

$n=2$: $\circledast \left\{ \begin{array}{l} \text{regular} \\ \text{algebraic} \\ \text{cusp. auto.} \\ \text{rep}^{s} \text{ of } GL_2(\mathbb{A}) \end{array} \right. \overset{\oplus}{\longleftrightarrow}$ cuspidal (elliptic) modular forms

He wants to explain the meaning of $\circledast$, & how $\oplus$ comes about.

He won't really say anything about $n>2$.

Survey of results / what he wants to talk about

By $S_k(\Gamma_1(N))$ we mean cusp forms on $\Gamma_1(N)$ of weight k, - a typical elt has the form $f : \mathfrak{h} \to \mathbb{C}$.

For $n>0$ $\exists$ Hecke operator $T_n : S_k(\Gamma_1(N)) \circlearrowleft$

The T_n commute. (They were talked about in USA last term)

Call f an **eigenform** if $f | T_n = c_n(f) f \quad \forall n$.

Facts

① $f(z) = \lambda \sum_{n=1}^{\infty} c_n(f) e^{2\pi i n z}$

② $E_f = \mathbb{Q}(\{c_n(f)\})$ is a number field

③ $S_k(\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}} S_k(\Gamma_0(N), \chi)$

and $f \in S_k(\Gamma_0(N), \chi)$ for some χ.

There are too many eigenforms to make any sense of Langlands' philosophy.

Quotient out then: write $\underset{S_k(\Gamma_1(N))}{\underset{\cap}{f}} \sim \underset{S_k(\Gamma_1(M))}{\underset{\cap}{g}} \iff c_p(f) = c_p(g)$ for all but finitely many $\underset{\text{primes}}{p}$.

Fact: If $f \sim g$ then $c_n(f) = c_n(g)$ $\forall n$ s.t. $(n, NM) = 1$.

In each ~-equiv. class $\exists!$ (up to scalar) f of lowest level, say N.

Then

1) If $f \sim g$, & g of level M, then $N | M$

& 2) If $N | M$ then $\exists g$ of level M with $g \sim f$.

f is said to be a newform (possibly after normalisation - notations differ)

Now fix a prime l & fix embeddings $\overline{\mathbb{Q}} \subseteq \mathbb{C}$
$\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_l$ (eh ?)

First big result

If f is an eigenform of level N then

$\exists \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_l)$ s.t.

1) ρ cts (note : this makes sense!)

2) ρ factors through $\mathrm{Gal}(K/\mathbb{Q})$ where K is the max^l ext^n of $\mathbb{Q}$ unramified outside Nl
(note : composition of finite ext's of $\mathbb{Q}$ unramified outside Nl is also unr. outside Nl so it makes sense to define K; K is infinite-dim^l $/\mathbb{Q}$)

3) If $p \nmid N\ell$ then (note $\exists$ well-defined conjugacy class (Frob_p) in $\mathrm{Gal}(K/\mathbb{Q})$, and)

$$\begin{cases} \mathrm{tr}\,\rho(\mathrm{Frob}_p) = c_p(f) \\ \det \rho(\mathrm{Frob}_p) = p^{k-1}\chi(p) \end{cases}$$

*see later

note these make sense!

Note also: the Čebatorev density thm implies that the Frob_p are dense in the Gal gp, so ρ is determined (& in fact well overdetermined) by 3).

4) $\rho(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where c is complex conj^n: we say ρ is odd

5) ρ irred

6) If $p \mid N$, $p \nmid \ell$, then $\rho|_{D_p}$ can be described completely

7) If is a newform, then the conductor of ρ is N. (The conductor is easily defined away from ℓ - he's not quite sure of the state of the art at ℓ)

For Richard, the above 7 facts are a big reason why modular forms are so important.

Now given $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_\ell)$, we'll fudge ρ a bit.

$\exists$ val^n on $\mathbb{Q}_\ell$, extending !ly to a val^n on $\overline{\mathbb{Q}}_\ell$.

Write $v_\ell: \overline{\mathbb{Q}}_\ell^\times \to \mathbb{Q}$. Set $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ = ring of integers of $\overline{\mathbb{Q}}_\ell$

= elts of non-negative val^n

Then $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ = elts of non-neg. val^n

$\cup|$

$\mathfrak{m}_{\overline{\mathbb{Q}}_\ell}$ = elts of +ve val^n

$\mathfrak{m}_{\overline{\mathbb{Q}}_\ell}$ is not principal. $\mathcal{O}_{\overline{\mathbb{Q}}_\ell} / \mathfrak{m}_{\overline{\mathbb{Q}}_\ell} = \overline{\mathbb{F}}_\ell$.

It's not too difficult to find ρ s.t. in fact $\rho \to GL_2(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})$ & so we can do

$$\begin{tikzcd}
\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[dr, "\bar\rho"'] & GL_2(\mathcal{O}_{\overline{\mathbb{Q}}_\ell}) \arrow[d] \\
& GL_2(\overline{\mathbb{F}}_\ell)
\end{tikzcd}$$

$\overline{\mathbb{F}}_\ell$ has the discrete topology, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is cpct, & $\bar\rho$ is cts $\therefore$ im $\bar\rho$ is finite.

Facts about $\bar\rho$:

$\bar{1}$) $\bar\rho$ cts, im $\bar\rho$ finite

$\bar{2}$) $\bar\rho$ factors thro' $\mathrm{Gal}(K/\mathbb{Q})$, K = max^l ext^n unr etc - but infact we can now make K a number field.

$\bar{3}$) ✓ mod l ie $\mathrm{tr}\,\bar\rho(\mathrm{Frob}_p) = c_p(f)$ mod l etc

$\bar{4}$) ✓ $\bar\rho(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\bar{5}$) X : $\bar\rho$ may be reducible.

$\bar{6}$) nothing

$\bar{7}$) If f is a newform, $\mathrm{cond}(\bar\rho) \mid N$

What about $GL_2(\mathbb{A})$? How does this tie in?

Call a rep^n ρ or $\bar{\rho}$ <u>modular</u> if it arises in this way.

Say ρ or $\bar{\rho}$ is <u>modular of level N wt k</u> if it arises from a <u>newform</u> f of level N wt k.

↑ may be eigenform depending on notation. Richard will stick with Newform.

<u>Conj 1</u> (Mazur) If $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_\ell)$ is cts, odd, irred, unramified outside a finite set of primes, & a certain cond^n on $\rho|_{D_\ell}$ holds, i.e. ρ is potentially semi-stable with Hodge-Tate numbers 0 & $k-1$, $k \geq 1$, then ρ is modular of wt k & level cond(ρ).

$\exists$ other conjectures of this form.

Cnj 1 for one ℓ implies STW conj.

<u>Conj 2</u> (Serre) If $\bar{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_\ell)$ is cts, odd, & irred then $\bar{\rho}$ is modular.

The potentially semistable stuff isn't mentioned in conjecture 2, but it does come in when you ask about the wt & level.

Conj 2 $\forall \ell \Rightarrow$ STW too

Remarks

1) For GL_1, the analogues are true, by class field theory. He may well explain this.

2) If $\bar{\rho} : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_3)$ then conj 2 is true.

This is Langlands' result: get a wt 1 form then increase wt.

Questions 1) What are the possible wts & levels in conj 2?

2) What about mod l rep's?

3) What about the rel^n ship between conj 1 & 2?

What RT will do in the course (perhaps) (unless people get bored / lost)

lots of quoting of results

1) Generalities about l-adic / mod l rep's of Galois gps & class field theory

2) Elliptic modular forms & associating$^{ed?}$ l-adic rep's

proofs here

3) Relationship between modular forms in the classical setting and in adèlic language - tricky but not profound. This would Shows why this generalises class field theory. also a very convenient language when he comes to talk about Shimura curves

4) Question 1 above. Lots seems to be known. In particular he'll prove Ribet's thm on lowering the level. This last seems to be the most difficult bit.

Rks on Ribets thm / Q1

Ribet is coming from Berkeley to give a seminar on Wednesday, on question 1. If you understand the talk you may give up the course.

Assume $l > 2$, $k \geq 2$. (for some reason)

Thm If $\bar{\rho}$ is modular then it's modular of level coprime to l.

Serre gives a recipe, given $\bar{\rho}\big|_{I_l}$, to get $k(\bar{\rho}) \in \mathbb{Z}$, $k(\bar{\rho}) \geq 2$.

Thm (Gross - Edixhoven) If $\bar{\rho}$ is modular of level N, coprime to l, & wt $k \geq 2$, then

1) $k \geq k(\bar{\rho})$
2) $k \equiv k(\bar{\rho}) \bmod (l-1)$
3) (hardest bit) $\bar{\rho}$ is also modular of wt $k(\bar{\rho})$ & level dividing N (with Richards def of modular wt level.)

If $\bar{\rho}\big|_{I_l}$ is trivial, then $\bar{\rho}$ should be "modular of wt 1"

Gross & Edixhoven have proved this in most cases

NB $S_1(\Gamma_1(N), \bar{\mathbb{F}}_l)$ has a modular def.

$S_k(\Gamma_1(N), \bar{\mathbb{F}}_l) = S_k(\Gamma_1(N), \mathbb{Z}) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_l$ but I think he said that another def was needed / useful in wt 1.

Given $\bar{\rho}$, Serre also constructs $\chi_{\bar{\rho}}$:

take $(\det \bar{\rho})\,\omega^{2-k(\bar{\rho})}$ ② , 1 & lift to char zero. Take $\chi_{\bar{\rho}}$ = ! lift with min^{l} conductor.

Thm (Carayol / Serre) If $\bar{\rho}$ is modular of level N, prime to l, & wt k, & if $l > 3$, then $\bar{\rho}$ is modular of level dividing N, wt k, & character $\chi_{\bar{\rho}}$.

It may be a bit more sensible to put $k(\bar{\rho})$ instead of k. It's OK though as $k(\bar{\rho}) \equiv k$ mod $l-1$.

Richard ~~thinks~~ is sure that the thm is also true if we replace k by $k(\bar{\rho})$ (as these are equivalent)

Thm (Ribet et al) If $\bar{\rho}$ is modular of level N & wt 2 then it is modular of wt 2 & level cond $(\bar{\rho})$ (away from l)

This should be true in any wt $k \geq 2$.
RT doesn't know the exact status of the thm for $k \geq 2$. Jordan-Livné announced a pf based on joint work with Faltings which hasn't appeared yet. Ribet may well have 2 proofs but doesn't want to publish them for some reason - possibly because he doesn't want to tread on Jordan & Livné's toes.

Crucial case (Ribet) if $\bar{\rho}$ comes from a newform $f \in S_2(\Gamma_1(N) \cap \Gamma_0(p))$ & if $\bar{\rho}$ is unramified at p, then f is modular of level dividing N & wt 2.

RT may well end up by proving this hopefully, following Ribet.

Summary of facts (essential for the language) on...

① l-adic rep's

a) Galois groups

L/K Galois $\iff$ L/K algebraic, normal & separable

Gal(L/K) = gp of autos of L which fix K

Gal(L/K) has a natural topology - the minimal topology s.t. Gal(L/K) is a top. gp (ie $\times$ & $^{-1}$ cts) & s.t. Gal(L/M) is an open subgp for M/K finite. ③

$\therefore$ L/K finite $\Rightarrow$ Gal(L/K) discrete

$$\mathrm{Gal}(L/K) = \varprojlim_{\substack{M/K \text{ finite Galois} \\ M \subseteq L}} \mathrm{Gal}(M/K), \text{ with maps } N \supseteq M : \mathrm{Gal}(N/K) \overset{\text{res}}{\twoheadrightarrow} \mathrm{Gal}(M/K)$$

The inverse limit is in the category of top. gps. ④

This implies that Gal(L/K) is cpct & totally disconnected. In fact, it's profinite ⑤

FTG: Intermediate fields between L & K $\longleftrightarrow$ Closed subgps of Gal(L/K)

see eg Gruenberg's article in C&F, I guess.

$$\text{by} \quad M \longmapsto \mathrm{Gal}(L/M)$$
$$\underset{\substack{= \text{fixed field} \\ \text{of } H.}}{L^H} \longleftarrow\!\shortmid\; H$$

As usual, M/K Galois $\iff$ Gal$(L/M) \trianglelefteq$ Gal(L/K) & in this case, Gal$(M/K) \cong$ Gal(L/K) / Gal(L/M).

Various references for this stuff: Washingtons bk on cyclotomic fields has an appendix on it. It's not really profound - in fact it's not an unreasonable (but long) exercise to try & prove all this stuff from a knowledge of finite Galois thy. In Langs book on Algebra, it's done in the exercises.

A crucial example: K a finite field.

$\forall n \in \mathbb{Z}_{\geq 1}$ $\exists!$ extn K_n/K of degree n, & K_n/K is Galois, with cyclic Galois gp, generated by a canonical generator Frob which satisfies $\mathrm{Frob}(x) = x^{\# K}$

If $\bar{K}$ = alg cl of. K, look at $\mathrm{Gal}(\bar{K}/K) = \varprojlim_n \mathrm{Gal}(K_n/K)$

$\cong \varprojlim_n \mathbb{Z}/n\mathbb{Z}$

What are the maps in this inverse system?

Fix notation s.t. $\mathrm{Frob} \in \mathrm{Gal}(K_n/K)$

$\begin{matrix} \updownarrow & & \updownarrow \\ 1 & \in & \mathbb{Z}/n\mathbb{Z} \end{matrix}$

⑥

Then the maps turn out to be:, if $m | n$, $\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$.

$\varprojlim_n \mathbb{Z}/n\mathbb{Z} =_{df} \hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ (easy exercise) ⑦

In $\mathrm{Gal}(\bar{K}/K)$ $\exists$ canonical Frobenius elt, as the Frobenii are compatible, & in fact it corresponds to $(1,1,-) \in \prod_p \mathbb{Z}_p$ ⑧

Note Frob $\in \mathrm{Gal}(\overline{K}/K)$ generates an infinite cyclic gp $\mathrm{Frob}^{\mathbb{Z}} \subseteq \mathrm{Gal}(\overline{K}/K)$

$$\begin{tikzcd}
\mathrm{Gal}(\overline{K}/K) \arrow[r, phantom, "\cong"] & \hat{\mathbb{Z}} & \varprojlim_n (r \bmod n) \\
\mathrm{Frob}^{\mathbb{Z}} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\cong"] & \mathbb{Z} \arrow[u, phantom, "\cup"] & r \arrow[u, mapsto]
\end{tikzcd}$$

⑨

Exercise - this diagram commutes, $\mathrm{Frob}^{\mathbb{Z}}$ is dense in $\mathrm{Gal}(\overline{K}/K)$, $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$,

& so if we're talking about cts maps, it suffices to understand the restriction to $\mathrm{Frob}^{\mathbb{Z}}$

6) Local fields Reference: Serre - Local fields.

Say $K/\mathbb{Q}_p$ is finite (ie non-arch local field of char zero)

$v_K : K^\times \twoheadrightarrow \mathbb{Z}$ a discrete valuation. Want to understand Gal gps. of local fields.

Reminder: if L/K alg: $\exists!\ v_K : L^\times \to (\mathbb{Q}, +)$ val^n (may not be discrete if L/K infinite)

⑩ $\mathcal{O}_L$ = integral closure of $\mathcal{O}_K$ in L

$= \{x \in L \mid v_K(x) \geq 0\}$ $(v_K(0) = +\infty)$

$\mathcal{O}_L$ is a local ring ~~&~~ - it has a ! max^l ideal, $\mathfrak{p}_L$,

where $\mathfrak{p}_L = \{x \in L \mid v_K(x) > 0\}$

We have a residue field $k_L = \mathcal{O}_L/\mathfrak{p}_L$, a field, & an algebraic ext^n of $k_K = \mathcal{O}_K/\mathfrak{p}_K$, a finite field, ⑪ by def of local field

If L/K is finite (eg $L=K$) then $\mathfrak{p}_L$ is principal, & we'll write $\mathfrak{p}_L = (\pi_L)$, π_L a typical generator.

A subclass of particularly useful extensions is...

L/K is <u>unramified</u> $\iff \mathfrak{p}_L = \pi_K \mathcal{O}_L$

(1.2) $\overset{\text{trivial}}{\iff} v_K(L^\times) \overset{!}{=} \mathbb{Z}$

$\iff$ L/K Galois & $\mathrm{Gal}(L/K) \xrightarrow{\text{natural map}} \mathrm{Gal}(k_L/k_K)$ is ~~sur~~injective

hence unramified ext's are always cyclic

Facts:

- The compositum of 2 unramified ext's is unramified ⑬
- $L'/L/K$, L'/L unr, L/K unr $\Rightarrow L'/K$ unr.

(Hence given any ext^n $\exists$ max^l unramified subext^n.)

L/K is <u>totally ramified</u> if $k_L = k_K$.

If L/K is any alg ext^n $\exists!$ intermediate field M, the max^l unramified subext^n, s.t. M/K is unramified & L/M is totally ramified. ⑭

If L/K is Galois then $I_{L/K} \overset{\text{df}}{=} \mathrm{Gal}(L/M)$ is called the inertia subgp of $\mathrm{Gal}(L/K)$. ⑮

Take $L = \overline{K}$: K^{nr}/K is max^l non-ramified ext^n

$$\begin{tikzcd} \text{Then } \mathrm{Gal}(K^{nr}/K) \arrow[r, phantom, "\cong"] & \mathrm{Gal}(\overline{k_K}/k) \arrow[r, phantom, "\cong"] & \hat{\mathbb{Z}} \\ \mathrm{Frob} \arrow[u, phantom, "\in" rotate=90] & \mathrm{Frob} \arrow[l, dashed] \arrow[u, phantom, "\in" rotate=90] & \end{tikzcd}$$

Frob defined here as image.

So we're left to study $I_{\overline{K}/K} = \mathrm{Gal}(\overline{K}/K^{nr})$.

Set $W_K = \{\sigma \in \mathrm{Gal}(\overline{K}/K) \mid$ image of σ in $\mathrm{Gal}(K^{nr}/K)$ lies in $\mathrm{Frob}^{\mathbb{Z}}\}$

$W_K \subseteq \mathrm{Gal}(\overline{K}/K)$

$$\begin{tikzcd}
1 \arrow[r] & I_{\overline{K}/K} \arrow[r] \arrow[d, equal] & W_K \arrow[r] \arrow[d, phantom, "\cap \text{ dense ⑯}"] & \mathrm{Frob}^{\mathbb{Z}} \arrow[r] \arrow[d, phantom, "\cap \text{ dense}"] & 1 \\
1 \arrow[r] & I_{\overline{K}/K} \arrow[r] & \mathrm{Gal}(\overline{K}/K) \arrow[r] & \mathrm{Gal}(K^{nr}/K) \arrow[r] & 1
\end{tikzcd}$$

W_K is the <u>Weil gp</u>. It's given the topology s.t. $I_{\overline{K}/K}$ is open with its usual top.

$$\begin{tikzcd}
G \arrow[d, dash] & K \\
D \arrow[d, dash, "f"'] & K \text{ in this case} \\
I \arrow[d, dash] & K^{nr} \\
1 & L
\end{tikzcd}$$

Rk: if $u \in \mathcal{O}_{K^{nr}}^{\times}$ & $(m,p)=1$, then $\sqrt[m]{u} \in K^{nr}$ ⑰

Now assume that L/K is Galois, & $I_{L/K}$ is finite

$$\begin{tikzcd}
L \arrow[d, dash, "\text{finite}"] \\
M \arrow[d, dash, "\text{non-ramified}"] \\
K
\end{tikzcd}$$

Then $\mathfrak{p}_L = (\pi_L)$, ie $\exists\, v_L : L^{\times} \to \mathbb{Z}$ discrete val^n ⑱

$$(v_L = (\# I_{L/K}) \times v_K)$$

Define $I_{L/K,i} =_{df} \{\sigma \in I_{L/K} \mid \sigma\pi_L/\pi_L \equiv 1 \ (\mathfrak{p}_L^i)\}$, $i \geq 1$

<u>exercise - indpt of choice of π_L</u> ⑲

The $I_{L/K,i}$ are the <u>higher ramification groups</u>

Elementary rks about them:

$I_{L/K,i} \lhd \mathrm{Gal}(L/K)$

$I_{L/K,i} = \{1\}$ if $i \gg 0$. ⑳

$I_{L/K} / I_{L/K,1} \hookrightarrow k_L^{\times}$

via $\sigma \longmapsto \frac{\sigma \pi_L}{\pi_L}$ exercise: 1) indpt of π_L
2) Homomorphism

& if $i \geq 1$, $I_{L/K,i} / I_{L/K,i+1} \hookrightarrow \mathfrak{P}_L^{i} / \mathfrak{P}_L^{i+1} \cong (k_L, +)$ (a p-group)

non-canonical - divide by π_L^{-i}.

via $\sigma \longmapsto \frac{\sigma \pi_L}{\pi_L} - 1$

ex 1) (indpt of π_L)
2) (homomorphism)

So $I_{L/K,1}$ is the ! Sylow p-subgp of $I_{L/K}$ (! as $I_{L/K,1} \lhd I_{L/K}$)

Also, $I_{L/K}$ is soluble.

If $I_{L/K}$ vanishes, recall L/K is unramified.

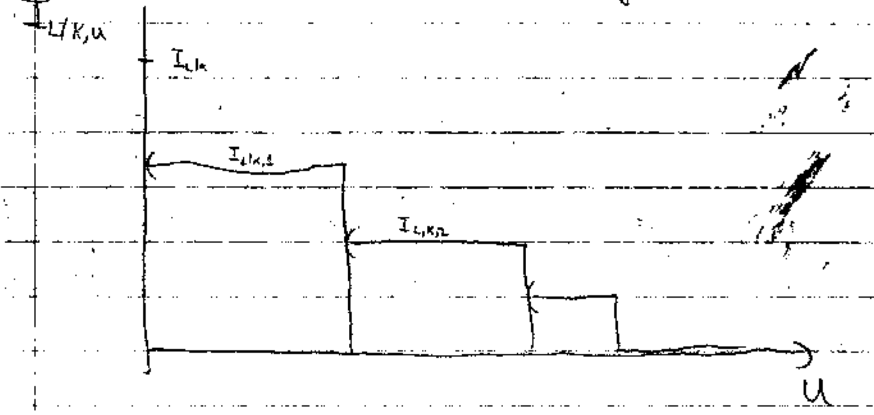
If $I_{L/K,1} = \{1\}$, we say L/K is tamely ramified.

If not, it's wild.

It turns out that there's 2 good ways of numbering these higher ramification gps, both useful. This is what we'll look at next.

Define, for $u \in [0,\infty)$,

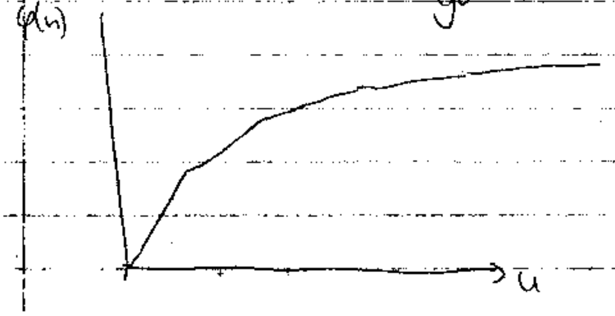
$$I_{L/K,u} = \begin{cases} I_{L/K} & \text{if } u=0 \\ I_{L/K,i} & \text{if } u>0 \text{ \& } i \text{ is the smallest integer } \geq u \end{cases}$$



Set $g_i = \# I_{L/K,i}$ ($I_{L/K,0} = I_{L/K}$ I guess)

Now define $\varphi : [0,\infty) \to [0,\infty)$, strictly increasing,

by $\varphi(u) = \frac{1}{g_0} \cdot (g_1 + \dots + g_i + (u-i) g_{i+1})$ if $u \in [i, i+1]$. (well-defined @ endpt.)



$\varphi(0)=0$, & φ is increasing, a bijection $[0,\infty) \to [0,\infty)$

Define $I^{u}_{L/K} = I_{L/K,\varphi^{-1}(u)}$ for $u \in [0,\infty)$

The upper numbering is useful when we extend L, namely...

Prop^n If $L'/L/K$, L'/K, L/K Galois, $I_{L'/K}$ finite,

then under $\mathrm{Gal}(L'/K) \twoheadrightarrow \mathrm{Gal}(L/K)$,

$$I^u_{L/K} = \text{image of } I^u_{L'/K} \quad \square$$

/ pf eg CaF p38 in case L'/K finite.

So upper numbering is good for quotients (ext^ns of L)

Lower numbering is good for subgps (exts of K) (by def^n!) ②
- this last ↑ is easy - this however needs some work - see e.g. Serres book.

Because the upper numbering works well under restriction, we can take inverse limits &, ...

if L/K is any Galois ext^n, set

$$I^u_{L/K} = \varprojlim_{\substack{L'/K \\ L' \subseteq L \\ I_{L'/K} \text{ finite}}} I^u_{L'/K} \qquad \forall u \in [0, \infty)$$

u is called a *jump* for L/K if $I^u_{L/K} \supsetneqq I^{u+\varepsilon}_{L/K}$ $\forall \varepsilon > 0$.

eg CaF p157 (cor 1 to thm)

Thm (Hasse-Arf) If L/K is an abelian ext^n then the jumps all occur at integers $\square$

False if L/K not abelian, even if L/K finite - upper numbering messes up jumps. Of course, if L/K finite then lower numbering jumps are obviously at integers.

Rk L/K is tame $\Leftrightarrow I^u_{L/K} = \{1\}\ \forall u > 0$ [illegible] ㉒

Rk The compositum of 2 tame ext's is tame (easy exercise) ㉓

Note - if L/K tame $\overset{\text{then}}{}$ $(\# I_{L/K}, p) = 1$ as $I_{L/K} \hookrightarrow k_L^\times$
- this does exercise above, I guess.

If L/K is any Galois ext^n $\exists!$ max^l tame subext^n.

Let K^t/K denote the max^l tame ext^n.

$$\begin{tikzcd}
\overline{K} \arrow[d, dash, "\text{pro-}p\text{-gp}"] \\
K^t \arrow[d, dash] \\
K^{nr} \arrow[d, dash, "\hat{\mathbb{Z}}"] \\
K
\end{tikzcd}$$

pro-p-gp (exercise) (ie profinite gp all of whose finite quotients are p-gp's) ㉔

In case of local fields $\text{Gal}(K^t/K)$ can be given pretty explicitly.

The bit $\text{Gal}(\overline{K}/K^t)$ is still quite mysterious.

Consider an ext^n L/K^{nr}, tame, of degree m.
(we're trying to understand $\text{Gal}(K^t/K)$)

We know $\underset{\text{cpct}}{\text{Gal}(L/K^{nr})} \hookrightarrow \underset{\text{discrete}}{\overline{k}_K^\times}$ ㉕

$\therefore$ finite image $\therefore$ cyclic image

㉖

Hence L/K^{nr} tame Galois degree m $\Rightarrow$ $\mathrm{Gal}(L/K^{nr})$ is cyclic, & $(m,p)=1$.

- p15, note ⑰ As it happens, we've remarked that K^{nr} contains all m^{th} roots of units if $(m,p)=1$ - in ptic, all m^{th} roots of unity if $(m,p)=1$.

Hence $\mathrm{Gal}(L/K^{nr})$ cyclic, $(m,p)=1$ $\Rightarrow L = K^{nr}(\sqrt[m]{\alpha})$

(p90 lemma 2 of C&F) for some $\alpha \in K^{nr}$ by Kummer thy / Hilbert 90.

$$= K^{nr}(\sqrt[m]{\pi_K^r}) \qquad (\alpha = \pi_K^r u,\ u \in \mathcal{O}_{K^{nr}}^{\times},$$

$$= K^{nr}(\sqrt[m]{\pi_K}) \qquad \therefore \sqrt[m]{u} \in \mathcal{O}_{K^{nr}}^{\times})$$

$\subseteq$ is obvious: then look at degrees!

$\Rightarrow \mathrm{Gal}(L/K^{nr}) \cong \mu_m$, the gp of m^{th} roots of 1.

$$\sigma \mapsto \frac{\sigma\sqrt[m]{\pi_K}}{\sqrt[m]{\pi_K}}$$ (Kummer theory again) (see eg. C&F p90 lemma 1)

So K^{nr} has a ! tame ext^n of any degree m coprime to p, given by $K^{nr}(\sqrt[m]{\pi_K})$ - check this is tame (exercise)
(obvious as it has degree m prime to p - ptc)

$$\text{Hence } \mathrm{Gal}(K^t/K^{nr}) = \varprojlim_{\substack{m \\ (m,p)=1}} \mathrm{Gal}(K^{nr}(\sqrt[m]{\pi_K})/K^{nr})$$

$$= \varprojlim_{\substack{m \\ (m,p)=1}} \mu_m$$

What are the maps? If $m|n$, $\mu_n \twoheadrightarrow \mu_m$
$J \mapsto J^{n/m}$ ㉖

These are the transition maps.

$\varprojlim_{\substack{m \\ (m,p)=1}} \mu_m \cong \varprojlim_{m} \mathbb{Z}/m\mathbb{Z}$ but this is non-canonical - it depends on choices of m^{th} roots of 1.

Let's assume we choose compatible primitive m^{th} roots of unity ζ_m for $(m,p)=1$, i.e. if $m|n$, $\zeta_n^{n/m} = \zeta_m$.

Get $\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ (reduction) if $m|n$

So the inverse limit $\underset{\text{exercise}}{\cong} \prod_{l \neq p} \mathbb{Z}_l$. (pretty obvious really)

So we have

$$\begin{tikzcd}
\overline{K} \arrow[d, dash, "\text{pro-}p\text{-gp}"] \\
K^t \arrow[d, dash, "\cong \prod_{l\neq p} \mathbb{Z}_l \ \text{(non-canonical)}"] \\
K^{nr} \arrow[d, dash, "\hat{\mathbb{Z}} \ni \text{Frob}"] \\
K
\end{tikzcd}$$

$$\begin{tikzcd}
 & \prod_{l\neq p}\mathbb{Z}_l \arrow[d, phantom, "\cong"] & & \hat{\mathbb{Z}} \arrow[d, phantom, "\cong"] & \\
0 \arrow[r] & Gal(K^t/K^{nr}) \arrow[r] & Gal(K^t/K) \arrow[r] & Gal(K^{nr}/K) \arrow[r] & 1 \\
 & & & \text{Frob} \arrow[u, phantom, "\in"] &
\end{tikzcd}$$

So there is an action of $\hat{\mathbb{Z}}$ on $\prod_{l\neq p}\mathbb{Z}_l$ (as latter is abelian)

by, if $\sigma \in \hat{\mathbb{Z}}$
$\tau \in \prod_{l\neq p}\mathbb{Z}_l$, $\sigma(\tau) = \tilde{\sigma}\tau\tilde{\sigma}^{-1}$

where $\tilde{\sigma} \in Gal(K^t/K)$ restricts to σ

Check indpt of $\tilde{\sigma}$ (as $Gal(K^t/K^{nr})$ abelian) (trivial)

& defines an action of $\hat{\mathbb{Z}}$ on $\prod_{l\neq p}\mathbb{Z}_l$.

What is this action? If we know how Frob acts, we know how $\mathrm{Frob}^{\mathbb{Z}}$ acts, & so because everything's cts we know exactly how $\hat{\mathbb{Z}}$ acts.

Fact: Frob acts on $\prod_{l \neq p} \mathbb{Z}_l$ via mult^n by $\# k_K$.

<u>Pf outline</u> (ie exercise)

- it's enough to treat L/K^{nr} a finite tame ext^n. Then $\mathrm{Gal}(L/K^{nr})$ is iso to μ_m & STP:

Frob acts on $\mathrm{Gal}(L/K^{nr}) \cong \mu_m$ by $\zeta \mapsto \zeta^{\# k_K}$

To prove this, observe

① $\mathrm{Gal}(L/K^{nr}) \xrightarrow{\sim} \mu_m$ preserves the action of $\mathrm{Gal}(K^{nr}/K)$
($\mu_m \subseteq K^{nr}$)

(27)

② Frob acting on ζ is $\zeta^{\# k_K}$ if ζ is a root of 1 of order prime to p. (we certainly have a congruence mod a prime above p by def^n - so STP no 2 distinct roots are congruent).

20/1/92 <u>All the above was for K local ie $K/\mathbb{Q}_p$ finite.</u>

That's all we can say about it really. He doesn't think this decomposition idea will give us any new info.

<u>Another idea: $\mathrm{Gal}(\overline{K}/K)^{ab}$ = max^l cts ab. image</u>
$= \mathrm{Gal}(\overline{K}/K) / [\mathrm{Gal}(\overline{K}/K), \mathrm{Gal}(\overline{K}/K)]$

Then

$$\begin{tikzcd}
 & \overline{K} \arrow[dl, dash] \arrow[d, dash] \\
K^{ab} \arrow[dr, dash] & K^{t} \arrow[d, dash] \\
 & K^{nr} \arrow[d, dash] \\
 & K
\end{tikzcd}$$

(NT)

㉓

Also (check)

$$\begin{tikzcd}
\mathrm{Gal}(\bar{K}/K) \arrow[r, two heads] & \mathrm{Gal}(K^{ab}/K) \arrow[r, two heads] & \mathrm{Gal}(K^{nr}/K) \\
W_K \arrow[u, phantom, "\cup\ \text{dense}"] \arrow[r, two heads] & W_K^{ab} = W_K/\overline{[W_K, W_K]} \arrow[u, phantom, "\cup\ \text{dense}"] \arrow[r, two heads] & \mathrm{Frob}^{\mathbb{Z}} \arrow[u, phantom, "\cup\ \text{dense}"]
\end{tikzcd}$$

㉘ (p^{15})

<u>Thm</u>

1) $\exists r_K : K^\times \xrightarrow{\sim} W_K^{ab} \subseteq \mathrm{Gal}(K^{ab}/K)$ ㉙

$\quad\quad\quad\quad\quad\quad \cup$

$r_K \mathcal{O}_K^\times = I_{K^{ab}/K}$ $\quad (= \mathrm{Gal}(K^{ab}/K^{nr})$ - see p144 of C+F)

$r_K (1 + \mathfrak{p}_K^i) = I^i_{K^{ab}/K}$, $i \in \mathbb{Z}_{\geq 1}$

$r_K \pi_K \in \mathrm{Frob}\, I_{K^{ab}/K}$

Warning - could have had $r_K \pi_K \in \mathrm{Frob}^{-1} I_{K^{ab}/K}$.

Richard thinks that his notation is not standard.
Another point of confusion is that some people define Frob to be our $(\mathrm{Frob})^{-1}$.

2) L/K finite $\quad W_L^{ab} \downarrow W_K^{ab}$ $\quad$ Then

$$\begin{tikzcd}
L^\times \arrow[r, "r_L", "\sim"'] \arrow[d, "\text{Norm } N"'] & W_L^{ab} \arrow[d] \\
K^\times \arrow[r, "r_K", "\sim"'] & W_K^{ab}
\end{tikzcd}$$

$(W_L \hookrightarrow W_K)$

p141 C+F

And

$$\begin{tikzcd}
L^\times \arrow[r, "r_L", "\sim"'] & W_L^{ab} \\
K^\times \arrow[u, hook] \arrow[r, "r_K", "\sim"'] & W_K^{ab} \arrow[u, "\text{transfer map}"']
\end{tikzcd}$$

p141 C+F

(cf. $H < G$ of finite index
$\exists$ transfer map $G^{ab} \to H^{ab}$)

If $\sigma \in \mathrm{Gal}(\overline{K}/\mathbb{Q}_p)$

$$\begin{tikzcd} K^\times \arrow[r, "r_K", "\sim"'] \arrow[d, "\sigma", "\wr"'] & W_K^{ab} \arrow[d] \\ (\sigma K)^\times \arrow[r, "r_{\sigma K}", "\sim"'] & W_{\sigma K}^{ab} \end{tikzcd}$$

conj by σ:

$$\begin{tikzcd} W_K \arrow[d] & \tau \arrow[d, mapsto] \\ W_{\sigma K} & \sigma\tau\sigma^{-1} \end{tikzcd}$$

(30)

- we can't check these as we don't know def of r !!

Note : $W_K^{ab} \cong K^\times \Rightarrow$ 1d rep's of W_K = 1d rep's of $K^\times$.

We now have an excellent description of $\mathrm{Gal}(K^{ab}/K)$

$\rho_0 : W_K \to GL_n(E)$ E a field, ρ_0 cts w.r.t. discrete topology on E.

$I_{\overline{K}/K}$ is cpct, $I_{\overline{K}/K} \subseteq W_K$ (31) so $I = \rho_0 I_{\overline{K}/K}$ is finite

'filtrand''
i^{th} filtrand is I_i

$$\begin{tikzcd} L \arrow[d, dash, "I"] \\ K^{nr} \arrow[d, dash] \\ K \end{tikzcd}$$

~~I guess $I_i = \rho_0 I^i_{\overline{K}/K}$~~ NO! (32)

- I is finite $\therefore \exists$ lower numbering on $I_{L/K}$ & $I_i = I_{L/K,i}$

, the more ramified it is, the bigger f is

Define the Artin conductor $f(\rho_0) =_{df} \sum_{i=0}^{\infty} \frac{1}{[I:I_i]} \underbrace{\dim(V/V^{I_i})}_{(=0 \text{ for } i \gg 0)}$ (33)

A Weil-Deligne rep^n $(\rho_0, N) : W_K \to GL_n(E)$ is

1) a rep^n $\rho_0 : W_K \to GL_n(E)$ cts. w.r.t. disc. top. on E

2) $N \in M_{n\times n}(E)$ is nilpt.

s.t. 3) $\rho_0(\sigma) N \rho_0(\sigma)^{-1} = \|\sigma\|^{-1} N$ where

$\|\cdot\| : W_K \twoheadrightarrow \mathrm{Frob}^{\mathbb{Z}} \to \mathbb{Q}^\times$ $\quad$ ($\mathrm{Frob} \mapsto (\# k_K)^{-1}$)

(NT)

Again $\exists \pm 1$ problem. He believes what he'll do is consident but he doesn't guarantee it'll be consistent with anything else.

Later today he'll give 2 reasons why these Weil-Deligne rep's are the correct things to look at.

Defⁿ The conductor of (ρ_0, N) is

$$f(\rho_0, N) = f(\rho_0) + \dim\left(V^{I_{\bar{K}/K}} / (\ker N)^{I_{\bar{K}/K}}\right) \quad (34)$$

eg $N=0$: $f(\rho_0, 0) = f(\rho_0)$

The class of Weil-Deligne rep's strictly contains the class of usual rep's

Defⁿ (ρ_0, N) is called F-semi-simple if ρ_0 is semisimple. ie direct sum of irred.

People talk about Weil-Deligne gps - a cute little extⁿ of W_K which is a gp scheme. We will call ρ_0, N a W-D repⁿ although this notion is the same as a repⁿ of the W-D gp.

See eg Tate p 20 defⁿ 4.1.1

Lemma Assume char $E = 0$, possibly. If (ρ_0, N) is a WD repⁿ, & if we set $q_K = \# k_K$, then
(35) $\exists!$ unipotent $u \in GL_n(E)$ s.t. $(\rho_0 u^{\log_{q_K} \|\ \|}, N)$ is an F-ss WD repⁿ of W_K. This latter object is called the F-semisimplification of (ρ_0, N).

A good reference for all this stuff is Tates article in Corvalis conference ed. by Borel & Casselman. This book is called automorphic rep's of something & L-fⁿs or sthg.

(Try Artrep2 Deligne)

ℓ-adic rep^s

$E/\mathbb{Q}_\ell$ finite.

$\rho : \mathrm{Gal}(\overline{K}/K) \to GL_n(E)$, cts w.r.t. the ℓ-adic topology on E.

These occur in nature. eg:

- Tate modules of ell curves or ab. varieties
- ℓ-adic cohomology of any variety $/K$
- finite image reps

Arith. Alg. Geom - v. important in

Principal interest is really global fields, but to understand global fields you localise.....

Assume $\ell \neq p$. Then "ρ can't be too complicated"

- namely fix $\varphi \in W_K$ s.t. $\varphi \mapsto \mathrm{Frob}$ in map $W_K \to \mathrm{Frob}^{\mathbb{Z}}$

Fix $t : \mathrm{Gal}(K^t/K^{nr}) \to (\mathbb{Q}_\ell, +)$ nontrivial (! up to mult by a scalar) (36)

Prop^n (Grothendieck)

1) If $\rho : \mathrm{Gal}(\overline{K}/K) \to GL_n(E)$ is an ℓ-adic rep^n then $\exists$ a ! W-D rep^n (ρ_0, N) $W_K \to GL_n(E)$ s.t.

$$\rho(\varphi^n \sigma) = \rho_0(\varphi^n \sigma)\exp(Nt(\sigma)) \quad \forall \sigma \in I_{\overline{K}/K},\ n \in \mathbb{Z}.$$

Moreover, the eigenvalues of $\rho_0(\varphi)$ are ℓ-adic units.

2) The iso. class of (ρ_0, N) is indpt of the choice of φ & t.

(ρ_0, N) is F-ss $\iff$ $\rho(\varphi)$ is ss. (ie diagonalisable) to $GL_n(E)$, $E/\mathbb{Q}_\ell$ finite!

3). For any WD rep^n (ρ_0, N) s.t. the evals of $\rho_0(\varphi)$ are ℓ-adic units, $\exists!$ ℓ-adic rep^n ρ which gives rise to (ρ_0, N) as above. The iso. class of ρ does not depend on t or φ.

Pts 2) & 3) are elementary - even an exercise. (37)
Pt 1) is not difficult. If you look at the sketch pf in Tates article it then becomes an exercise, if a long one.
Frazer says it took RT ≥ 1 hr to prove it in his pt III course a couple of yrs ago.

Def^n The conductor $f(\rho)$ of ρ is defined to be $f(\rho_0, N)$

Now another litany of def^ns (with the occasional statement)

Local Langlands

E a field, V/E a possibly infinite-dim^l v.s., $K/\mathbb{Q}_p$ finite.

A rep^n $\pi : GL_n(K) \to \mathrm{Aut}_E(V)$ is called admissible if

1) If $U \subseteq GL_n(K)$ is an open subgp then V^U is a f.d. v.s.

smooth & 2). If $v \in V$ then $\mathrm{stab}_{GL_n(K)}(v)$ is open

Again a strange def^n - we'll just have to wait & see why this is a useful concept.

Usually ∞-dim^l things get attacked by an analytical techniques - but 1) & 2) give finiteness cond^ns to give an algebraic flavour to the idea.

Examples

n=1 (Exercise)

π an irred. admissible rep of $GL_1(K) = K^\times$ $\overset{(38)}{\Rightarrow}$ $\dim V = 1$

$\pi : K^\times \to E^\times$

π admissible $\Leftrightarrow$ ~~π~~ cts w.r.t. disc. top on $E^\times$ (39)

LCFT $\longrightarrow$ $\updownarrow$ (40)

$\pi^G : W_K \to E^\times$ cts w.r.t. disc. top on $E^\times$

$(\pi^G, 0)$ is a WD-repⁿ of W_K. (note $n=1 \Rightarrow N=0$)

Assuming a statement of CFT, $\exists$ bijection

$$\left\{\begin{matrix}\text{irred. admiss}\\ \text{rep's of } GL_1(K)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{1-dim}^{\ell}\ \text{WD}\\ \text{rep}^{n}\text{s of } W_K\end{matrix}\right\}$$

$$\pi \longmapsto (\pi^G, 0)$$

$$\chi^A \longleftarrow (\chi, 0)$$

$= \chi^{\text{Admissible}}$

(more explicitly: $\chi^A : K^\times \xrightarrow[\sim]{r_K} W_K^{ab} \xrightarrow{\chi} E^\times$)

Conductor $f(\pi)$ = the smallest r s.t. $\pi|_{1+p_K^r}$ is trivial (maybe $r=0$)

ie $\dim V^{\pi(1+p_K^r)} > 0$

Here $1 + p_K^0 = \mathcal{O}_K^\times$ (!!)

Then $f(\pi) = f(\pi^G, 0)$ (41)

He apologises for the lack of mathematics in these lectures. He just thinks it'd be nice to see def's now & see them in action later.

n=2 He's gonna restrict to $\overset{\overset{E}{\shortparallel}}{\mathbb{C}}$.

Say $\chi_1, \chi_2 : K^\times \to \mathbb{C}^\times$ admissible.
(any old maps not ness. anything)

$I(\chi_1, \chi_2) = \{ \varphi : GL_2(K) \to \mathbb{C} \mid \varphi$

- φ locally cst (ie $g \in GL_2(K) \Rightarrow \exists\, U$ open, $g \in U$, φ cst on U
- $\varphi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b) \left|a/b\right|^{1/2}_{\cancel{K}} \varphi(g)$ $\}$

Here $|x|_K = (\# k_K)^{-v_K(x)}$ as usual.
$\in \mathbb{R}_{\geq 0}$

$\pi : GL_2(K) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(I(\chi_1, \chi_2))$

$(\pi(h)\varphi)(g) = \varphi(gh)$ (42)

Exercise $I(\chi_1, \chi_2)$ is admissible

Facts about ~~admissible~~ reps $I(\chi_1, \chi_2)$

① $I(\chi_1, \chi_2)$ is admissible

② $I(\chi_1, \chi_2)$ is irred if $\chi_1/\chi_2 \neq |\cdot|_K^{\pm 1}$ (43)

③ $\chi_1/\chi_2 = |\cdot|_K^{-1} \Rightarrow \exists$ an exact sequence

(44) $0 \to (\chi_1 |\cdot|_K^{1/2}) \circ \det \to I(\chi_1, \chi_2) \to S(\chi_1, \chi_2) \to 0$

1-dim'l admissible & $S(\chi_1, \chi_2)$ is irred.

④ ~~If~~ $\chi_1/\chi_2 = |\cdot|_K \Rightarrow \exists$

$0 \to S(\chi_2, \chi_1) \to I(\chi_1, \chi_2) \to (\chi_2 |\cdot|_K^{1/2}) \circ \det \to 0$

same S as in ③

note ⑤ is vacuous

⑤ If L/K is quadratic, $\chi: L^\times \to \mathbb{C}$ is admissible & if $\chi \circ \sigma \neq \chi$ (where $\mathrm{Gal}(L/K) = \{1, \sigma\}$) then $\exists$ irred admiss. rep^n $BC_L^K(\chi)$ of $GL_2(K)$

⑥ $I(\chi_1, \chi_2)$, $S(\chi_1, \chi_1 |\cdot|_K)$, $BC_L^K(\chi)$ are all ∞-dim^l irred. admiss.

⑦ The only iso between the things in ⑥ is

$$I(\chi_1, \chi_2) \cong I(\chi_2, \chi_1) \text{ if } \chi_1/\chi_2 \neq |\cdot|_K^{\pm 1}$$

References: Gelbart - Automorphic rep's on adèle gps (sketches)

Jacquet - Langlands book (unreadable) ($\exists$ pfs)

⑧ If char $k_K \neq 2$ then these are all the ∞-dim^l irred. admiss rep's of $GL_2(K)$

Rk: (in k_K any char): the only finite-dim^l ones are of the form $\chi \circ \det$ where $\chi: K^\times \to \mathbb{C}^\times$ is admissible

Exercise: ⑨ If π is an admissible irred rep^n of $GL_2(K)$ then the centre $K^\times$ acts on π by an admissible char $\chi_\pi : K^\times \to \mathbb{C}^\times$ (45) the <u>central characte</u>

Exercise (46) $\chi_{I(\chi_1, \chi_2)} = \chi_1 \chi_2$

Exercise (47) $\chi_{S(\chi_1, \chi_2)} = \chi_1 \chi_2$

I think Exercise (48) $\chi_{\varphi \circ \det} = \varphi^2$

not an exercise as ∄ def of BC

$\chi_{BC_L^K(\varphi)} = \varphi|_{K^\times} \cdot \varepsilon_{L/K}^A$ where

$$\varepsilon_{L/K}^A : K^\times \twoheadrightarrow K^\times / NL^\times \longrightarrow \{\pm 1\} \subset \mathbb{C}^\times$$

~~2:33~~ 2:27 ⑩ π ∞-dim^{l}, irred, admissible.

$$U_1(\underset{\mathcal{O}_K}{\overset{\cap}{a}}) = \{ g \in GL_2(\mathcal{O}_K) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \ (a) \}$$

= open, cpct subgp of $GL_2(K)$ (49)

$\exists f(\pi) \in \mathbb{Z}_{\geq 0}$ st. $\dim \pi^{U_1(a)} = \max(0, 1 + v_K(a) - f(\pi))$ (50)

= conductor = min^{l} integer r st. $\pi^{U_1(\pi_K^r)} \neq 0$.

<u>Def</u> π is <u>unramified</u> if$_{df}$ $f(\pi) = (0)$ $(\Leftrightarrow \pi^{GL_2(\mathcal{O}_K)} \neq (0))$

see note 51 for pf. → $f(I(\chi_1, \chi_2)) = f(\chi_1) + f(\chi_2)$ $\quad \chi_1/\chi_2 \neq |\cdot|_K^{\pm 1}$

$$f(S(\chi, \chi|\cdot|_K)) = \begin{cases} 1 & f(\chi) = 0 \\ 2f(\chi) & f(\chi) \neq 0 \end{cases}$$

$$f(\underset{\text{base change}}{BC_L^K}(\chi)) = \begin{cases} 2f(\chi) & \text{if } L/K \text{ is unramified} \\ f(\chi) + f(\varepsilon_{L/K}^{A}) & \text{if } L/K \text{ ramified} \end{cases}$$

1-dim^{l}: $f(\chi \circ \det) =_{df} 2f(\chi)$

⑪ <u>Local Langlands $n=2$</u>

$$\exists \text{ bijection } \left\{ \begin{matrix} \text{irred. admiss.} \\ \text{rep's of } GL_2(K) \\ \text{over } \mathbb{C} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} F\text{-ss WD reps} \\ (\rho, N) : W_K \to GL_2(\mathbb{C}) \end{matrix} \right\}$$

$$\pi \longmapsto \pi^{G}$$

2:35 ~~2:4~~ s.t. 1) $f(\pi) = f(\pi^G)$

2) $(\chi_\pi)^G = \det(\pi^G)$

– not so difficult if $p \neq 2$ as we have classification above: I, S, BC.

Restrict to $\mathbb{C}$ & recall:

$V/\mathbb{C}$; $K/\mathbb{Q}_p$ finite.

$\pi : GL_n(K) \to \mathrm{Aut}_{\mathbb{C}}(V)$ is admissible $\overset{}{\Leftrightarrow}_{df}$ 1) $U \subseteq GL_n(K)$ open $\Rightarrow \dim V^U < \infty$
2) $v \in V \Rightarrow \{g \in GL_n(K) \mid gv = v\}$ open

Local CFT: $\left\{ \begin{matrix} \text{irred admiss rep's} \\ \text{of } GL_1(K) \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{1-dim}^l \text{ W-D} \\ \text{rep}^s \text{ of } W_K \end{matrix} \right\}$:

$\pi \longmapsto$ [illegible]

$\chi^A \longleftarrow\!\!| \ \chi$

Now $GL_2(K)$: irred. admiss. $I(\chi_1, \chi_2)$ for $\chi_1/\chi_2 \neq |\cdot|_K^{\pm 1}$
$S(\chi, \chi|\cdot|_K)$

L/K quadratic, $\chi : L^\times \to \mathbb{C}^\times$
$\chi \circ \sigma \neq \chi$, $1 \neq \sigma \in \mathrm{Gal}(L/K)$

$BC_L^K(\chi)$
$\chi \circ \det$

If res char $\neq 2$, this is all irred rep's.

Recall

⑪ $\exists$ bijection $\left\{ \begin{matrix} \text{irred admiss.} \\ \text{rep's of } GL_2(K) \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{2-dim}^l\text{, F-ss} \\ \text{WD rep's of } W_K \end{matrix} \right\}$

$= (\rho, N) : W_K \to GL_2(\mathbb{C})$

(easy as they've got same cardinality! $2^{\aleph_0}$)

$\pi \longmapsto \pi^G$

$\pi_{(\rho,N)} \longleftarrow\!\!| \ (\rho, N)$

st $f(\pi) = f(\pi^G)$
central char of π $\rightarrow (\chi_\pi)^G = \det \pi^G$ } not so easy!

(NT)

NB if you fix $f(\pi)$ & χ_π $\exists$ only finitely many π corresponding to this, so ⑪ has a lot of content.

(We can do even better than this by fixing L & ε- factors)

Rk: ⑪ is true for GL_n $\forall n$ (~~Giyenigat~~ Guy Henniart) ($\exists$ notion of conductor)
(- we haven't defined $f(\pi)$ if $\pi : GL_n(K) \to \mathrm{Aut}(V)$, $n>2$)

Rk: in fact we can define L & ε factors for both sides, & these should match. Known for $n=1,2,3$, & conjectured $\forall n$.

This is the local Langlands conjecture. Recently proved $\forall n$ over function fields.

Now back to ⑪. The correspondence:

π	π^G	N
$I(\chi_1, \chi_2)$	$\begin{pmatrix}\chi_1^G & 0 \\ 0 & \chi_2^G\end{pmatrix}$	$N=0$
"special rep" $\to$ $S(\chi, \chi\lvert\cdot\rvert_K)$	$\begin{pmatrix}\chi^G & 0 \\ 0 & \chi^G\lvert\cdot\rvert_K^G\end{pmatrix}$	$N=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ (check a WD rep)
$BC_L^K(\chi)$	$\mathrm{Ind}_{W_L}^{W_K}(\chi^G)$ (irred)	$N=0$
$\chi_0 \det$	$\begin{pmatrix}\chi^G(\lvert\cdot\rvert_K^{1/2}) & 0 \\ 0 & \chi^G(\lvert\cdot\rvert_K^{1/2})^G\end{pmatrix}$	$N=0$

↑ all irred admiss of vector $\neq 2$

note: L/K so $\chi : L^\times \to K^\times$ $\mathbb{C}^\times$?! , $\chi^G : W_L \to \mathbb{C}^\times$

$W_L \subset W_K$ index 2

(as
$$\begin{tikzcd} 0 \arrow[r] & I_{\bar K/L} \arrow[r] \arrow[d, "\iota"] & W_L \arrow[r] \arrow[d, "r"] & \mathbb{Z} \arrow[r] \arrow[d, "2\cdot"] & 0 \\ 0 \arrow[r] & I_{\bar K/K} \arrow[r] & W_K \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$
)

degenerate case of I when $\chi_1 = \chi_2 \lvert\cdot\rvert^{\pm 1}$

Exercise 1) check conductors on RHS & central chars & check they all work. Conductor of $\mathrm{Ind}_{W_L}^{W_K}(\chi^G)$ is easy if you know about inducting conductors if not $\exists$ bare hands method.

(ST)

Exercise 2) Check that if char $k \neq 2$ these actually are all the F-ss 2-d WD reps of W_K (52)

May have to know the classifications of finite subgps of $PGL_2(\mathbb{C})$: ab, dih, A_4, S_4, A_5

NB $\exists$ (due to Bushnell $\&\sim$) classification of irred admiss for $n > 2$ (any char)
- bijection is still tricky as ε etc is tricky.
It's only primes $\leq n$ that ~~may~~ cause a problem.

Richards feeling with this local conjecture is that its hard but people have a lot of ideas & progress is continually being made: it's just a case of clever people getting on with it. In the global case we're still thrashing around in the dark.

Rk π admiss rep of $GL_2(K)$

$U, U' \subseteq GL_2(K)$ open cpct subgps
$g \in GL_2(K)$

Hecke operator: $[UgU'] : \pi^{U'} \to \pi^{U}$
$$x \mapsto \sum_{i=1}^{r} g_i x$$

where $UgU' = \coprod_{i=1}^{r} g_i U'$.

exercise: 1) Check $\begin{matrix} U \text{ open}, U' \text{ cpct} \\ (\text{or other way round}) \end{matrix} \Rightarrow r$ finite (53)
2) well-defined.

He wants to define 2 particular Hecke operators:

$$T = [GL_2(\mathcal{O}_K)\begin{pmatrix} \pi_K & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathcal{O}_K)]$$

$$S = [GL_2(\mathcal{O}_K)\begin{pmatrix} \pi_K & 0 \\ 0 & \pi_K \end{pmatrix} GL_2(\mathcal{O}_K)]$$ (indpt of π_K) (54)

(exercise!) - certainly if π irred then only interesting if π unr.

Only time the action is interesting is when π unramified irred

he claims to have never said this.
lets restrict to this case anyway.

π irred unr, $\dim \pi^{GL_2(\mathcal{O}_K)} = 1$

$$T, S : \pi^{GL_2(\mathcal{O}_K)} \to \pi^{GL_2(\mathcal{O}_K)}$$

$\therefore$ get scalars $\pi(T), \pi(S) \in \mathbb{C}$.

The point of doing this is that he wants to tell us what $\pi(T)$ & $\pi(S)$ are in terms of the earlier nonsense

χ_1, χ_2?

uniformiser

$$\pi = I(\chi_1, \chi_2),\ \chi_1/\chi_2 \neq |\cdot|_K^{\pm 1} \Rightarrow \pi(S) = \text{central char} = \chi_\pi(\pi_K)$$
$$\pi(T) = (\# k_K)^{\frac{1}{2}} (\chi_1(\pi_K) + \chi_2(\pi_K))$$

Exercise (55) (hard unless ...)

HINT: ~~$B(K)\,GL_2(R)$~~ $B(K)\,GL_2(\mathcal{O}_K) = GL_2(K)$

$\overset{\shortparallel}{\text{upper}\,\Delta} \in GL_2(K)$

identify $\pi^{GL_2(\mathcal{O}_K)}$ with this hint.

S is easy;

$$T:\ GL_2(\mathcal{O}_K)\begin{pmatrix}\pi_K & 0\\ 0 & 1\end{pmatrix}GL_2(\mathcal{O}_K) = \coprod_{\alpha \in \mathcal{O}_K/\mathfrak{p}_K} \begin{pmatrix}\pi_K & \alpha\\ 0 & 1\end{pmatrix} GL_2(\mathcal{O}_K)$$

(is left!)

$$\amalg \begin{pmatrix}1 & 0\\ 0 & \pi_K\end{pmatrix} GL_2(\mathcal{O}_K)$$

(- just copy computation for $SL_2(\mathbb{Z})$!!)

That finishes local fields

c) Numberfields

$K/\mathbb{Q}$ finite $\Leftrightarrow_{df}$ K is a no. field

A prime of K $=_{df}$ d.v. on K

eg $K=\mathbb{Q}$ $\quad$ p a prime of $\mathbb{Z}$

$$\updownarrow$$

v_p disc. val^n $v_p(p^r\, m/n)$: $(mn, p)=1$, $r\in\mathbb{Z}$

- this is all discrete val's of $\mathbb{Q}$ (thm) (Ostowski - see eg C+F p[illegible])

These are in bijection with the non-zero prime ideals of O_K:

$$\left(\alpha\in K^\times : \ (\alpha) = \mathfrak{p}^r \prod \mathfrak{q}_i^{s_i} ; \ v_{\mathfrak{p}}(\alpha) = r \right)$$

Given a prime v we can form the completion of K at v, K_v, a local field, & look at O_{K_v}, $\mathfrak{p}_{K_v}$, $k_v = O_{K_v}/\mathfrak{p}_{K_v}$

- if $v \leftrightarrow \mathfrak{p}$, $k_v = O_K/\mathfrak{p}$.

eg $K=\mathbb{Q}$, $k_{v_p} = \mathbb{F}_p$.

If L/K algebraic, v a prime of K, then $\exists$ valns w of L s.t. $w|_K = v$. We write $w|v$.

If L/K is Galois, then all the extensions w are conjugate via $G(L/K)$. This easily implies that if L/K is finite $\exists$ only finitely many such extensions, (& they'll all be discrete $\therefore$ primes of L)

We want an analogue of the completion of L - must be careful if L/K infinite.

$\exists$ a field $L_{(w)}/K_v$ algebraic & an embedding

$$\begin{tikzcd} L \arrow[rr, hook] & & L_{(w)} \\ & K \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

s.t. $L_{(w)} = L.K_v$

& the ! valⁿ on $L_{(w)}$ extending v restricts to L to give w.

— this is a roundabout way of saying things.

Another way:

if L/K is finite, $L_{(w)} = L_w$.

The reason things get convoluted is that an infinite extⁿ of a local field may not be complete & we don't want $L_{(w)}$ to be complete in this case.

$$\begin{tikzcd} \overline{\mathbb{Q}_p} \arrow[d, dash] & \supseteq & \overline{\mathbb{Q}} \arrow[d, dash] \\ \mathbb{Q}_p & \supseteq & \mathbb{Q} \end{tikzcd}$$

but

$$\begin{tikzcd} \widehat{\overline{\mathbb{Q}_p}} & & \\ \cup & & \\ \overline{\mathbb{Q}_p} \arrow[d, dash] & \supseteq & \overline{\mathbb{Q}} \arrow[d, dash] \arrow[uull, dash, "\cancel{\text{algebraic}}"'] \\ \mathbb{Q}_p & \supseteq & \mathbb{Q} \end{tikzcd}$$

completion ✓

not algebraic

Certainly $LK_v \subseteq L_{(w)}$

So really $L_{(w)} = \{x \in L_w \mid x \text{ alg} / K_v\}$. — indpt of w I guess

If L/K Galois $w|v$ we say L/K is unramified at $v \Leftrightarrow L_{(w)}/K_v$ unr.

Fact If L/K is finite, it's unr at all but finitely many primes

Now consider L/K Galois, $w|v$. Set $D_w = \{\sigma \in \mathrm{Gal}(L/K) \mid w \circ \sigma = w\}$

~~W~~ $\mathcal{D}_w = \{\sigma \in \mathrm{Gal}(L/K) \mid w\circ\sigma = w\}$

-the <u>decomposition gp</u>, a closed subgp.

Ex: $\mathcal{D}_{w\circ\sigma} = \cancel{\sigma\mathcal{D}_w\sigma^{-1}}\ \sigma^{-1}\mathcal{D}_w\sigma$ $\left(\tau\in D_w \Leftrightarrow w\tau = w \Leftrightarrow w\sigma\sigma^{-1}\tau\sigma = w\sigma \Leftrightarrow \sigma^{-1}\tau\sigma\in D\right)$

$w\circ\sigma\tau = w\circ\sigma$ iff $\sigma\tau\sigma^{-1}\in D_w$

$\mathcal{D}_w \xrightarrow{\cong} \mathrm{Gal}(L_{(w)}/K_v)$. $\left(\xrightarrow{\text{iso} \Leftrightarrow \text{unr}} \mathrm{Gal}(k_{L_{(w)}}/k_{K_v})\right)$ (56)

In ptic, if L/K is unr at v then $\exists$ Frob $\in \mathrm{Gal}(L_{(w)}/K_v)$ - gives an elt of D_w.

Given v, we thus get a conjugacy class (as any $w|v$ gives D_w & all D_w conjugate) $(\mathrm{Frob}_v) \subseteq \mathrm{Gal}(L/K)$

<u>Rk</u> Say L/K is finite. Then what <u>is</u> (Frob_v)? Pick $w|v$. It's the conjugacy class of the ! $\sigma\in\mathrm{Gal}(L/K)$ s.t.

$$\sigma x \equiv x^{\#k_v} \bmod w \quad \text{for all } x\in\mathcal{O}_L.$$

$$x\equiv y\ (w) \text{ iff } w(x-y)\neq 0$$

$$\text{if } w \leftrightarrow \beta \text{ this is iff } x\equiv y\ (\beta)$$

Marcus

If S is any set of primes of K, & if L_1 & L_2 are 2 alg ext's of K unr. outside S then the compositum L_1L_2/K is <u>unramified</u> outside S & in ptic we can take the compositum of the lot & get K^S/K, max^l ext^n unr outside S.

<u>Thm</u>: Let S be a finite set of primes of K. Let $d\in\mathbb{Z}_{>0}$. Then $\exists$ only finitely many ext's L/K satisfying

1) $[L:K]\leq d$
2) L/K unr outside S.

We probably won't be using this thm.

We will be using...

Thm (Čebotarev) Let L/K be a finite Galois extⁿ.
Let $C \subseteq \mathrm{Gal}(L/K)$ be a conjugacy class.

Then $\exists$ infinitely many primes v of K, unramified in L, s.t. $(\mathrm{Frob}_v) = C$.
(in fact $\frac{|C|}{|G(L/K)|}$ = Dirichlet density of # such primes)

(57) Easy exercise: this implies the following useful form: Let S be a finite set of primes of K. Then $\mathrm{Gal}(K^S/K) \supseteq \bigcup_{v \notin S} (\mathrm{Frob}_v)$ & is dense

Note if we have sthg cts on $\mathrm{Gal}(K^S/K)$ it suffices to control it at all Frob_v, $v \notin S$.

Thm (Brauer–Nesbitt) Let G be a group & E a field.
Let $\rho_1, \rho_2 : G \to GL_n(E)$ be 2 semisimple repⁿs (= direct sum of irreds)

Assume ~~N_{ρ_1}~~ $\mathrm{tr} \Lambda^i \rho_1 = \mathrm{tr} \Lambda^i \rho_2$, $0 \le i \le n$.

Then $\rho_1 \simeq \rho_2$. (note $i=1$ good enough for char 0!)

Exercise – deduce this from rest of thm → If char $E = 0$ or $> n$ then we need only assume $\mathrm{tr}\, \rho_1 = \mathrm{tr}\, \rho_2$ (58)

$\mathrm{tr} \Lambda^i \rho g$ can be expressed in terms of $\mathrm{tr}\, g^j$, $j = 1, \dots, i$, with denominators $\le n$)

Rks 1) if $\alpha \in GL_n(E)$ then char poly of α is $\sum_{i=0}^{n} (-1)^i (\mathrm{tr} \Lambda^i \alpha) X^{n-i}$

so condⁿs of thm are equivalent to saying "$\forall g \in G$ char poly $\rho_1(g)$ = char poly $\rho_2(g)$"

2) ss means $\oplus$ irreds. ~~this is nothing if char 0 (ok?)~~

3) ρ any repⁿ, set $\rho^{ss} = \oplus$ Jordan Hölder constituents. Then ρ^{ss} is ss, & $\mathrm{tr} \Lambda^i \rho^{ss} = \mathrm{tr} \Lambda^i \rho \ \forall i$. (59)

Cor (of Br. Nes & Čeb)

Let K be a no. field, S a finite set of primes, of K, & let E be a topological field...

Let $\rho_1, \rho_2 : \mathrm{Gal}(K^S/K) \to GL_n(E)$ be 2 cts semisimple rep's

~~Assume $\mathrm{tr}\, \Lambda^i \rho_1 = \mathrm{tr}\, \Lambda^i \rho_2$ for $i = 0$...~~

Assume $\mathrm{tr}\, \Lambda^i \rho_1(\mathrm{Frob}_v) = \mathrm{tr}\, \Lambda^i \rho_2(\mathrm{Frob}_v)$ $\forall v \notin S$ $\forall i = 0 .. n$

↑
makes sense
as trace defined (exercise)
on conj class

Then $\rho_1 \cong \rho_2$. If char $E = 0$ or is $> n$, need only assume that $\mathrm{tr}\, \rho_1(\mathrm{Frob}_v) = \mathrm{tr}\, \rho_2(\mathrm{Frob}_v)$ $\forall v \notin S$

Pf Exercise ⑥⓪

Monday - finish this lot
Wed - Modular forms
Global CFT, l-adic chars etc

l-adic rep's

Say K is a number field, & $E/\mathbb{Q}_\ell$ finite

What is an ℓ-adic rep?

① A cts rep $\rho : \mathrm{Gal}(K^S/K) \to GL_n(E)$ (Here, S is a finite set of primes of K & K^S is max^l ext of K unramified outside S) (cts w.r.t. ℓ-adic topology on $GL_n(E)$)
(If we use discrete top then im ρ would be finite & so we're doing sthg much more general) is called an ℓ-adic rep on $GL_n(E)$

② ρ is <u>pure of wt w</u> if $\forall i : \overline{E} \hookrightarrow \mathbb{C}$ alg field embeddings (no topological restrictions) & all $v \notin S$ & all eigenvalues α of $\rho(\mathrm{Frob}_v)$ we have

$$|i(\alpha)| = (\#k_v)^{-w/2}$$

$\uparrow$ residue field

③ ρ is called <u>rational over</u> a number field $\underline{E_0} \subseteq E$ if $\forall v \notin S$, $\mathrm{tr}\, \Lambda^i \rho(\mathrm{Frob}_v) \in E_0$ $(\forall i)$

He guesses ② $\Rightarrow$ ③ - sthg like that

e.g. the archetypal example: if $A/\mathbb{Q}$ is an elliptic curve then $\underset{\text{Tate module}}{(T_\ell A)} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

- ℓ-adic rep
- rational over $\mathbb{Q}$
- $(\mathrm{tr}\, \mathrm{Frob}_p) = \# A(\mathbb{F}_p) - p - 1$
- pure of wt -1 (possibly +1 but he thinks -1)

(beginning of century)

More generally, if

V/K smooth (proper, but let's say) projective variety

$H^i_{\acute{e}t}(V \times_{\mathrm{spec}\, K} \mathrm{spec}\, \overline{K}, \mathbb{Q}_\ell) \circlearrowleft \mathrm{Gal}(\overline{K}/K)$

$\uparrow$ f.d. $/\mathbb{Q}_\ell$

French mathematicians working with Grothendieck, principally, worked all this out. It's kinda similar to Betti cohomology

Essentially the only source of ℓ-adic rep's there is.

- rational $/\mathbb{Q}$ (Grothendieck & his cronies)
- pure of wt i (Famous thm of Deligne)

Note the Tate module is the dual of $H^1_{ét}$ or sthg so we get -1 for wt.

Note for one i, one V_1 we get a whole bunch of l adic rep^s, one for each l.

④ Let $E_0/\mathbb{Q}$ be a number field.
By a compatible system of λ-adic rep^s $/E_0$ we mean:

prime = finite prime

place - could be infinite

i) for each finite prime $v \notin S$ a polynomial $Q_v(X) \in E_0[X]$

ii) for each prime λ of E_0 (with res char ℓ say), a cts rep ρ_λ $\mathrm{Gal}(\overset{S \cup \{\ell\}}{\bar{K}}/K) \to GL_r(E_{0,\lambda})$
s.t. $\rho_\lambda(\mathrm{Frob}_v)$ ($v \notin S$, $\underset{\text{ie } v \nmid \ell}{v \nmid \overset{\uparrow}{\underset{\text{norm of } \lambda}{N\lambda}}}$) has char poly $Q_v(X)$ (indpt of λ)

In ptic ρ_λ is rational (as $\mathrm{tr}\,\Lambda^i \rho_\lambda$ is $\pm$ coeff of X^i in $Q_v \in E_0$)

In each example above we have a compatible family $/\mathbb{Q}$.

⑤ A system $\{\rho_\lambda\}$ is called strongly compatible if for each prime v of K there's a W-D rep

$$(\rho_{v,0}, N_v) : W_{K_v} \to GL_r(E_0),$$

s.t. for each λ of res char $\neq$ res char of v we have

$$(\rho_{v,0}, N_v) \overset{\text{Grothendieck}}{\longleftrightarrow} \rho_\lambda \Big|_{\mathrm{Gal}(\bar{K}_v/K_v)} \quad \text{up to F-ss.}$$

note ⑤ ⇒ ④ in fact strongly compatible restricted to $v \notin S$ is ⇔ ④. Ⓒ

It's not known if $H^i_{ét}$ is strongly compatible.
He thinks it's true if $i=1$ (is been proved)
It's true for Tate mods for ell curves
It's conjectured that in this case we don't need to F-semisimplify.

Rk. If $\rho_\lambda : \mathrm{Gal}(K^{S\cup\{\ell\}}/K) \to GL_n(E)$ is a λ-adic rep^n, it can belong to at most one system of semi-simple λ adic rep^{n}s (exercise) (6i)

=compatible system

NB to say a global rep^n is ss is way different from saying restriction to local dec. gps is ss.

Lemma If $\rho : \mathrm{Gal}(\overline{K}/K) \to GL_n(E)$ is an ℓ-adic rep^n then
$\exists$ a finite free $\mathcal{O}_E$-module $\Lambda \subset E^n$ s.t. $\Lambda \otimes_{\mathcal{O}_E} E = E^n$, & s.t. $\Big\}$ is a lattice
Λ is preserved by $\rho(\sigma)$ $\forall \sigma \in \mathrm{Gal}(\overline{K}/K)$

In p^{tic} we get a reduction $\bar{\rho}_\Lambda : \mathrm{Gal}(\overline{K}/K) \to GL(\Lambda/m_E\Lambda)$

$$\overset{\shortparallel}{GL_n(k_E)}$$

If Λ' is another such lattice, then $\bar{\rho}_\Lambda^{ss} \cong \bar{\rho}_{\Lambda'}^{ss}$

If ρ is irred it doesn't follow that $\bar{\rho}_\Lambda$ irred.

Pf Let $\rho(\sigma) = (\rho(\sigma)_{ij})_{i,j = 1\ldots n}$ w.r.t. the standard basis $\{e_i\}$ of E^n.

Let $\Lambda_0 = \bigoplus \mathcal{O}_E e_i \subseteq E^n$ Λ_0 is a lattice
(ie Λ_0 is a finite free $\mathcal{O}_E$ mod s.t. $\Lambda_0 \otimes_{\mathcal{O}_E} E = E^n$)

image of Gal($\bar{K}$/K) cpt

$\{\rho(\sigma)_{ij} \mid \sigma \in \mathrm{Gal}(\overline{K}/K)\}$ is cpct so it's $\subseteq \mathfrak{m}_E^{-r_{ij}}$

By making r big enough, we can make it indpt of i,j

Then $\rho(\sigma)\Lambda_0 \subseteq \mathfrak{m}_E^{-r}\Lambda_0$

Set $\Lambda = \sum_{\sigma \in \mathrm{Gal}(\overline{K}/K)} \rho(\sigma)\Lambda_0$ This is certainly an $\mathcal{O}_E$-mod

Λ is certainly an $\mathcal{O}_E$-mod, & Λ is inv't under $\rho\,\mathrm{Gal}(\overline{K}/K)$.

Finally, $\Lambda_0 \subseteq \Lambda \subseteq \mathfrak{m}_E^{-r}\Lambda_0$ so Λ is a lattice in E^n with the required property. (as Λ f.g $\Rightarrow$ Λ lattice)

For the second bit, we have char poly $\bar\rho_\Lambda(\mathrm{Frob}_v)$ = char poly $\bar\rho_{\Lambda'}(\mathrm{Frob}_v)$

= reduction mod $\mathfrak{m}_E$ of char poly of $\rho(\mathrm{Frob}_v)$, for all but finitely many v.

so by Brauer-Nesbitt this implies the result:

$$\bar\rho_\Lambda^{ss} \cong \bar\rho_{\Lambda'}^{ss}$$

He now wants to tell us whats known in the subject of abelian l adic reps, & how this connects with global CFT.

~~By~~ K is a number field. By a <u>finite place</u> of K he means a prime of K ie a properly normalised discrete val^n.

An <u>infinite place</u> (or <u>∞-place</u>) is an equivalence class of embeddings $\sigma: K \to \mathbb{C}$

with $\sigma \sim c\circ\sigma$ = complex conjugation $\circ\,\sigma$.

Given an infinite place σ we get an absolute value $|\;|_\sigma$ on K given by $|x|_\sigma = |\sigma x|^{n_\sigma}$ where $n_\sigma = 1$ if $\sigma K \subseteq \mathbb{R}$

2 if not.

If v is an infinite place, then K_v is the completion of K by $|\cdot|_v$ ie if $v \leftrightarrow \sigma$, $K_v = \mathbb{R}$ if $\sigma K \subseteq \mathbb{R}$

$\mathbb{C}$ if not.

If $K_v = \mathbb{R}$ we call v <u>real</u>

$K_v = \mathbb{C}$ we call v <u>complex</u>

If v is a finite place set $| \; |_v : K_v^\times \to \mathbb{Q}^\times$
$x \mapsto (\# k_v)^{-v(x)}$

Then set $|0|_v = 0$

Define $\mathbb{A}_{K,\infty} = K_\infty = \prod_{\substack{v \text{ an} \\ \infty \text{ place of} \\ K}} K_v$

$\mathbb{A}_{K,\infty}$ is a topological ring (prod top) product of connected is connected.

$(K_\infty^\times)^\circ$ = connected cpt of 1 in $K_\infty^\times = \prod_{v \text{ complex}} \mathbb{C}^\times \times \prod_{v \text{ real}} \mathbb{R}^\times_{>0}$

$\mathbb{A}_K = \{ \underline{x} \in \prod_{\substack{v \text{ place} \\ \text{of } K}} K_v \mid x_v \in \mathcal{O}_{K_v}$ for all but finitely many $v \}$

This is the adèle ring - Rk $K \hookrightarrow \mathbb{A}_K$ diagonal as if $a \in K$, $(a) = \prod \mathfrak{p}_{v_i}^{a_{v_i}}$ + $v \nmid \mathfrak{p}_i \Rightarrow a \in \mathcal{O}_{K_v}$

We dont put the product topology on $\mathbb{A}_K$ as we want more open sets than this.

$\mathbb{A}_K$ is given ~~[illegible]~~ a topology st $\prod_{v \text{ finite}} \mathcal{O}_{K_v} \times K_\infty$ is open with its usual topology, & $\mathbb{A}_K$ is a top ring (63)

$\mathbb{A}_K$ is a large & unwieldy object when you first see it.

Set $\mathbb{A}_K^\infty = \mathbb{A}_{K,f}$ = same def without infinite places.

$\mathbb{A}_K = K_\infty \times \mathbb{A}_K^\infty$

$GL_n(\mathbb{A}_K)$ - we want this to be a topological gp

Don't consider $GL_n(\mathbb{A}_K) \subseteq \mathbb{A}_K^{n\times n}$ for topology:

- think of it as

$$GL_n(\mathbb{A}_K) \subseteq \{(d,g) \in \mathbb{A}_K \times M_{n\times n}(\mathbb{A}_K) \mid d \det g = 1\}$$

with subspace topology. Be careful!

Then $\prod_{\substack{v\nmid\infty \\ (\text{ie } v \text{ finite})}} GL_n(\mathcal{O}_{K_v}) \times GL_n(K_\infty)$ is an open subgp with the product topology. (64)

Rk $GL_n(K) \hookrightarrow GL_n(\mathbb{A}_K)$

In ptic, $n=1$: $GL_1(\mathbb{A}_K) = \mathbb{A}_K^\times$ = idèle gp

<u>Thm</u> (Gl CFT) (main thm) (- we have $\mathrm{Gal}(K^{ab}/K)$ & want to study it for K a number field)

1) $r_K : K^\times \backslash \mathbb{A}_K^\times / (K_\infty^\times)^\circ \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/K)$

$\uparrow$ $\mathbb{A}_K^\times$ is commutative so it doesn't matter if its left or right!

$\nwarrow$ not surprising, as $\mathrm{Gal}(K^{ab}/K)$ is totally disconnected!

Also $\mathrm{Gal}(K_v^{ab}/K_v) \hookrightarrow \mathrm{Gal}(K^{ab}/K)$ (65)

(its abelian!)

& in fact

$$\begin{tikzcd}
\mathbb{A}_K^\times / (K_\infty^\times)^\circ K^\times \arrow[r, "r_K", "\sim"'] & \mathrm{Gal}(K^{ab}/K) \\
r_{K_v} : K_v^\times \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{Gal}(K_v^{ab}/K_v) \arrow[u]
\end{tikzcd}$$

$\uparrow$ local CFT

commutative $\forall v$

2) L/K finite

$$\begin{tikzcd}
\mathbb{A}_L^\times \arrow[r] \arrow[d, "N\ (\text{exercise})\ (66)"] & \mathrm{Gal}(L^{ab}/L) \arrow[d] \\
\mathbb{A}_K^\times \arrow[r] & \mathrm{Gal}(K^{ab}/K)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{A}_K^\times \arrow[r] \arrow[d] & \mathrm{Gal}(K^{ab}/K) \arrow[d, "\text{transfer}"] \\
\mathbb{A}_L^\times \arrow[r] & \mathrm{Gal}(L^{ab}/L)
\end{tikzcd}$$

commutes

& of $\sigma \in \mathrm{Gal}(\bar{K}/\mathbb{Q})$

$$\begin{tikzcd}
\mathbb{A}_K \arrow[r] \arrow[d, "\sigma"'] & \mathrm{Gal}(K^{ab}/K) \arrow[d, "\text{conj by } \sigma"] & \tau \arrow[d, mapsto] \\
\mathbb{A}_{\sigma K} \arrow[r] & \mathrm{Gal}((\sigma K)^{ab}/\sigma K) & \sigma\tau\sigma^{-1}
\end{tikzcd}$$

eg $K=\mathbb{Q}$ $\quad$ $\mathbb{Q}^{ab} = \mathbb{Q}$(all roots of 1) $\quad$ (consequence of CFT, or thm, easier than CFT but proved using techniques of CFT)

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \varprojlim_N \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) = \varprojlim_N (\mathbb{Z}/N\mathbb{Z})^\times$$

$$\cong \hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times$$

$$= \hat{\mathbb{Z}}^\times \times \mathbb{R}^\times_{>0} / \mathbb{R}^\times_{>0} \qquad (!)$$

- note he writes $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ $\qquad$ $= \mathbb{A}^\times / \mathbb{R}^\times_{>0}\mathbb{Q}^\times$ (exercise)

- exercise: $\hat{\mathbb{Z}}^\times \times \mathbb{R}^\times_{>0} \cap \mathbb{Q}^\times = \{1\}$; $\mathbb{Q}^\times(\hat{\mathbb{Z}}^\times \mathbb{R}^\times_{>0}) = \mathbb{A}^\times$. (67)

So we get a map $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \longrightarrow \mathbb{A}^\times / \mathbb{R}^\times_{>0}\mathbb{Q}^\times$

& it's $r_{\mathbb{Q}}^{-1}$ unfortunately because of the way everything's been normalised. (68)

Because we want to talk about l-adic chars of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ & complex analytic chars of $\mathbb{A}^\times$, we develop the idea of

a Grossencharacter is a cts hom

$$\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times, \text{ cts w.r.t. standard top on } \mathbb{C}^\times$$

There are too many.

χ is called algebraic, or of type A_0 more classically,

if $\exists$ integers n_σ for $\sigma : K \hookrightarrow \mathbb{C}$

$$\chi\Big|_{(K_\infty^\times)^\circ}(x) = \prod_{\substack{v \text{ real} \\ \updownarrow \\ (\sigma : K \hookrightarrow \mathbb{R})}} x_v^{n_\sigma} \prod_{\substack{v \text{ complex} \\ \uparrow \\ \{\sigma, c\sigma\} : K \hookrightarrow \mathbb{C}}} (\sigma x_v)^{n_\sigma} (c \circ \sigma x_v)^{n_{c\circ\sigma}}$$

n.b. $K_v \cong \mathbb{R}$ canonically if v real
via σ

$K_v \cong \mathbb{C}$ but 2 choices for iso if v complex

e.g.

1) $\|\cdot\|$ $\quad x \longmapsto \prod_{v \text{ place}} |x_v|_v$.

Check it vanishes on $K^\times$ ✓ by product formula
$\|K^\times\| = 1$ (ex check for $\mathbb{Q}$) (69)

It's algebraic.

2) If $\chi_0 : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

$$\| \qquad\qquad\qquad$$

$\hat{\mathbb{Z}}^\times / U_N$ where $U_N = \{x \in \hat{\mathbb{Z}}^\times \mid x \equiv 1 \ (N)\}$

~~$\mathbb{A}^\times / U_N \mathbb{R}$~~ (see next pg)

$$\begin{tikzcd}
(\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\chi_0"] \arrow[d, phantom, "\|\wr"] & \mathbb{C}^\times \\
\hat{\mathbb{Z}}^\times/U_N \arrow[r, phantom, "\cong"] & \mathbb{A}^\times/U_N(\mathbb{R}^\times_{>0})\mathbb{Q}^\times \arrow[u, dashed, "\chi"']
\end{tikzcd}$$

$n_\infty = 0$. χ algebraic. ⑳

Weil tried to classify these things

Prop (Weil) Suppose χ algebraic. Then either 1) n_σ is indpt of σ
set $w_\chi = 2n_\sigma$

or 2) $K \supseteq L$, L a CM field
& $\exists w \in \mathbb{Z}$ & $\exists m_\sigma \, \forall \sigma : L \hookrightarrow \mathbb{C}$
s.t.
i) $m_\sigma + m_{c\sigma} = w$ indpt of σ
ii) $n_\sigma = m_{\sigma|_L}$

A CM field is a totally imaginary quadratic extn of a totally real field L^+
(totally real: $L^+ \hookrightarrow \mathbb{C} \Rightarrow L^+ \subseteq \mathbb{R}$)

(totally imag: $\sigma\, L \hookrightarrow \mathbb{C} \Rightarrow \sigma L \nsubseteq \mathbb{R}$)

((ie ∃! non-trivial elt which is
complex conjugation if L is CM a holds))

Alternative defn of CM

1) L is CM iff its normal closure is CM (min Galois ext of $\mathbb{Q} \supseteq L$)
2) If $L/\mathbb{Q}$ is Galois, then
L is CM iff $\exists c \in Gal(L/\mathbb{Q})$, $c \neq 1$
s.t. for all $\sigma : L \hookrightarrow \mathbb{C}$,
c is just complex $conj|_L$

It's 2:34

Recall: K a number field,

$$\chi : \underset{\text{(diag. embedding)}}{K^\times} \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times \quad \text{grossencharacter.}$$

It's algebraic if $$\chi\Big|_{(K_\infty^\times)^\circ}(x) = \prod_{\substack{\sigma : K \hookrightarrow \mathbb{R} \\ \updownarrow \\ v}} x_v^{n_\sigma} \prod_{\substack{v \text{ complex} \\ \updownarrow \\ \sigma,\, c\sigma}} (\sigma x_v)^{n_\sigma} (c\sigma x_v)^{n_{c\sigma}} .$$

The prop of Weil greatly restricts n

Recall our example: $\| \cdot \| = \prod_v | \cdot |_v$

& if χ_D is a Dirichlet char,

$$\chi_D : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$
$$\Big\}$$
$$\chi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$$

via $$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / U_N \mathbb{R}_{>0}^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times ,$$ [illegible].

Define $\pi_p \in \mathbb{A}_\mathbb{Q}^\times$ by $\pi_{p,v} = \begin{cases} 1 & v \neq p \\ p & v = p \end{cases}$

& ~~set~~ then $\chi_D(p) = \chi(\pi_p^{-1})$ (exercise). (71)

(given alg gross. χ):

Def$^=$: $\chi_0 : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$

$$\chi_0(x) = \chi(x) \prod_{\substack{v \text{ real} \\ \updownarrow \\ \sigma}} x_v^{-n_\sigma} \prod_{\substack{v \text{ complex} \\ \updownarrow \\ \{\sigma, c\sigma\}}} (\sigma x_v)^{-n_\sigma} (c\sigma x_v)^{-n_{c\sigma}}$$

- this rescales it at the infinite places to make it vanish.

Then in fact $\chi_0 : \mathbb{A}_K^\times / (K_\infty^\times)^\circ \to \mathbb{C}^\times$

In fact this construction, starting with χ an algebraic grossenchar, & ~~[illegible]~~ getting χ_0, is a bijection

$$\left\{\begin{matrix}\text{algebraic}\\ \text{grossenchars}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\chi_0 : \mathbb{A}_K^\times/(K_\infty^\times)^\circ \to \mathbb{C}^\times \\ \chi_0 \text{ cts} \\ \& \ \chi_0(\alpha) = \prod_{\sigma : K \hookrightarrow \mathbb{C}} (\sigma\alpha)^{-n_\sigma} \quad \forall \alpha \in K^\times \\ \text{[illegible]}\end{matrix}\right\}$$

(exercise) (72)

<u>Lemma</u>: $E_\chi =_{df} \mathbb{Q}(\text{image of } \chi_0)$. Then E_χ is a number field.

<u>Pf</u> Exercise. (73)

Let λ be a prime of E_χ, lying above l.

$$\begin{tikzcd} \text{Define } \chi_\lambda^G : \mathrm{Gal}(\overline{K}/K) \arrow[r] & \mathrm{Gal}(K^{ab}/K) \arrow[r, "\cong", "r_K\,(CFT)"'] & K^\times \backslash \mathbb{A}_K^\times / (K_\infty^\times)^\circ \arrow[d] \\ & & E_{\chi,\lambda}^\times \end{tikzcd}$$

where this ↓ map is

$$\begin{tikzcd} x \in K^\times \backslash \mathbb{A}_K^\times/(K_\infty^\times)^\circ \arrow[d, mapsto] \\ \chi_0(x)\, P_l(x_l) \end{tikzcd}$$

$\prod_{v|l} K_v^\times$ $\parallel$

where $x_l = (x_{v_1}, \dots, x_{v_r}) \in K_l^\times$, v_i primes above l

& $K^\times \to E_\chi^\times$

$x \mapsto \prod_{\sigma : K \hookrightarrow \mathbb{C}} (\sigma x)^{n_\sigma}$ has a ! cts ext $P_l : K_l^\times \to E_{\chi,\lambda}^\times$ (74)

(including real ones)

The reason we use algebraic grossenchars is we need to get from the infinite 'bit' of χ to sthg that makes sense at ℓ. (or sthg).

Exercise: $\{\chi_\lambda^G\}$ are a compatible system of λ-adic rep's which are rational $/E_\chi$ ⑮

eg. 1) $\chi = \|\cdot\|$, $E_\chi = \mathbb{Q}$, (exercise) (good one) χ_ℓ^G = inverse (he thinks) of the cyclotom. char. ⑯

where the cyclotomic char is this:

if K is a number field, union of $K(\zeta_{\ell^r})$, ζ_{ℓ^r} a prim ℓ^{r} rt [illegible]

$$\begin{tikzcd}
\chi_\ell : \mathrm{Gal}(\overline{K}/K) \arrow[r, two heads] \arrow[dr, "\chi_\ell"'] & \mathrm{Gal}(K(\zeta_{\ell^\infty})/K) \arrow[d] \\
& \mathbb{Z}_\ell^\times
\end{tikzcd}$$

Note $\ell \in \mathbb{Z}$
$K \neq \mathbb{Q}$ nec.

where, if $\sigma \in \mathrm{Gal}(\overline{K}/K)$, $\sigma\zeta_{\ell^n} = \zeta_{\ell^n}^{\chi_\ell(\sigma) \bmod \ell^n}$

This !ly defines a cts char, the cyclotomic char.

If $\mathfrak{p}$ is a prime of K, $\mathfrak{p} \nmid \ell$, then

$$\chi_\ell(\mathrm{Frob}\,\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$$

2) $\chi \leftrightarrow \chi_0 : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Q}(\zeta_{\varphi(N)})^\times \subseteq \mathbb{C}^\times$

& $E_\chi \subseteq \mathbb{Q}(\zeta_{\varphi(N)})$ &

I guess technically it should be $\xrightarrow{\sim} E_\chi^\times \subseteq E_{\chi,\lambda}^\times$

⑰

$(\text{st } \sigma(\zeta_N) = \zeta_N^{a_\sigma} \quad \overset{\sigma}{\underset{\cancel{a_\sigma}\, a_\sigma}{\downarrow}})$ via

$$\begin{tikzcd}
\chi_\lambda^G : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r, two heads] & \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \arrow[d, phantom, "\cong"] & & \\
& (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\chi_0^{-1}"] & \mathbb{Q}(\zeta_{\varphi(N)})^\times \subseteq \mathbb{Q}(\zeta_{\varphi(N)})_\lambda^\times
\end{tikzcd}$$

I think he said this

Rk: $\{\chi^G_\lambda\}$ are in fact strongly compatible. (may be exercise). (78)

Rk 1) χ^G_λ is pure of wt w, w as in prop^n (Weil)

2) $\chi^G_\lambda(\mathrm{Frob}_v) = \chi(\pi_v)$ for all but finitely many v, (79)

where $\pi_v \in \mathbb{A}_K^\times$ $(\pi_v)_w = \begin{cases} 1 & w \neq v \\ \underset{\text{at } v}{\text{unif}} & w = v \end{cases}$

3) χ^G_λ is Hodge-Tate ie if μ ~~is~~ a prime of K above ℓ,

then

completion alg cl

$$E_{\chi,\lambda} \otimes_{\mathbb{Q}_\ell} \widehat{\overline{K}_\mu}$$

(: inherits an action of $\mathrm{Gal}(\overline{K}_\mu/K_\mu)$)

linear action of $\mathrm{Gal}(\overline{K}_\mu/K_\mu)$ via χ^G_λ

semilinear natural action of $\mathrm{Gal}(\overline{K}_\mu/K_\mu)$

$$\cong$$

$$\bigoplus_i \widehat{\overline{K}_\mu}(a_i) \quad , a_i \in \mathbb{Z}$$

, $\mathrm{Gal}(\overline{K}_\mu/K_\mu)$ acts by twisting the usual semilinear action by $\chi_\ell^{a_i}$

Don't worry too much about this.

This statement only depends on $\chi^G_\lambda|_{I_{K_\mu}}$ for [illegible] $\mu | \ell$.

Thm (In Richard's mind, this is the main thm here. The set of chars arising from alg. grossenchars has several characterisations.)

Let $\psi : \mathrm{Gal}(\overline{K}/K) \to E^\times$ be an ℓ-adic rep^n ($E/\mathbb{Q}_\ell$)

Then TFAE:

1) ψ is Hodge-Tate (~~refers~~ refers to $\psi|_{I_{K_\mu}}$ for $\mu | \ell$)

2) ψ is rational (over ~~some~~ no field) (refers to $\psi|_{D_v}$ for good primes v)

3) ψ arises from some alg. grossenchar χ as above

Frazer proved this in his part III exam, although he "didn't understand a word of it".

In this case we also have

4) ψ is part of a compatible system (obvious as it's in $\{K_\lambda^\xi\}$ from 3)

5) ψ is pure (cf 1)

We call such ψ algebraic. (ie ψ satisfying 1, 2 or 3)

Pf is not too bad once you have CFT.
Then $3 \to 1$, $3 \to 2$; $1 \to 3$ if you understand Hodge Tate
immediately / ψ rational over E_χ ; is easy

$2 \to 3$ uses transcendence theory & is deeper.

He said "ℓ-adic cohomology of smooth proj varieties".

There is a generalisation of this: if $\psi : \mathrm{Gal}(\overline{K}/K) \to GL_2(E)$ (conjectural!!!)
then we can generalise 1, 2 & 3: 3 probably says
"ψ arises from a modular form" but none can prove
$1 \to 3$ or $2 \to 3$ in this case (in case $K = \mathbb{Q}$?)

<u>Lemma</u> If k is a finite field of characteristic l, & if $\chi : \mathrm{Gal}(\overline{K}/K) \to k^\times$ then there is an alg. char. $\tilde{\chi} : \mathrm{Gal}(\overline{K}/K) \to E^\times$ where $E/\mathbb{Q}_l$ is finite & $\mathcal{O}_E / \mathfrak{m}_E = k$, s.t. $\tilde{\chi} \bmod \mathfrak{m}_E = \chi$, & $f(\tilde{\chi}) = f(\chi)$.

(So any modular char arises from an alg char & in ptic from an alg grossenchar.)

In ptic $\exists$ alg grosschar X s.t.

$$X(\pi_v) \equiv \chi(\mathrm{Frob}_v) \quad (\lambda), \ \lambda | l$$

for almost all v, λ a prime of E_X

Also $f(X) = f(\chi)$ when we've defined $f(X)$ X a grossenchar (not had to define)

<u>Pf</u> Let $\overset{W}{\omega} : k^\times \to E^\times$ be the Teichmüller char, ie the ! char st ω W mod $\mathfrak{m}_E$ = id.

Let $\tilde{\chi} = \omega \circ \chi$. This works $\square$ (80)

For GL_n things are much much harder as $\nexists$ map $GL_2(k) \to GL_2(E)$

However there does appear to be some evidence that the lemma is true in higher dimensions.

This is the end of the first bit of the course

2/4/91 (snigger)

On 1pm on Tuesday next he expects us to have done lots of questions

2) Modular forms

In generalising the last bit, the case $n=2$, $K=\mathbb{Q}$ seems to be generalised by $\chi \rightsquigarrow$ cusp forms.

a) Basic facts Say $\mathfrak{h}$ = upper half complex plane

$$\text{If } f : \mathfrak{h} \to \mathbb{C} \ \& \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^{+}(\mathbb{R}) \ \& \ k \in \mathbb{Z}_{\geq 0}$$

det > 0

$$(f|_k \gamma) : \mathfrak{h} \to \mathbb{C} \text{ by } (f|_k \gamma)(z) = (\det \gamma)^{k-1} j(\gamma, z)^{-k} f(\gamma z)$$

$$\text{where } j(\gamma, z) = (cz+d) \quad \therefore j(\gamma\delta, z) = j(\gamma, \delta z) j(\delta, z)$$

$$\& \ \gamma z = \frac{az+b}{cz+d} \in \mathfrak{h} \text{ if } z \in \mathfrak{h}.$$

NB Shimura has $(\det \gamma)^{k/2}$ as then diagonal matrices act trivially

RT uses $k-1$ as it makes more sense in the adelic generalisation
Shimura has to twist by a determinant

If $\Gamma \subseteq SL_2(\mathbb{Z})$ of finite index

$M_k(\Gamma)$ = modular forms
$S_k(\Gamma)$ = cusp forms

- def's: $f : \mathfrak{h} \to \mathbb{C}$ are 1) holomorphic
2) $f|_k \gamma = f \quad \forall \gamma \in \Gamma$

& 3) if $\alpha \in SL_2(\mathbb{Z})$ then $(f|_k \alpha)(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z/N}$

$(N = N(\alpha) \in \mathbb{Z})$ for $\mathrm{Im}\, z \gg 0$

$M_k(\Gamma)$: $a_n = 0$ if $n < 0$
$S_k(\Gamma)$: $a_n = 0$ if $n \leq 0$.

When RT first tried to read Shimura's gilbk he was a bit mystified as to why the 2 classes of matrix gps

$$\Gamma_0(N) \trianglerighteq \Gamma_1(N)$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0(N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \cancel{SL_2(\mathbb{Z})}\, \Gamma_0(N) \;\middle|\; d \equiv 1(N) \right\}$$

were so important. It turns out that the adelic setting gives more motivation to these gps.

$$\Gamma_0(N) / \Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$$

~~$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod n$~~

Hence $\underset{\overset{\cup}{e}}{(\mathbb{Z}/N\mathbb{Z})^\times}$ acts on ~~$M_k(N)$~~ $S_k(\Gamma_1(N))$ or $M_k(\Gamma_1(N))$

e acts via $\langle e \rangle$ thus:

$$\text{if } \underset{\Gamma_0(N)}{\underset{\cap}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \mapsto e, \quad f|_k \langle e \rangle = f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

χ-eigenspace

$$\text{Then } \left.\begin{matrix} S \\ M \end{matrix}\right\}_k (\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} \left.\begin{matrix} S \\ M \end{matrix}\right\}_k (\Gamma_0(N), \chi)$$

(82)

he's never sure whether to write 0 or 1 there

$\begin{matrix} S \\ M \end{matrix}{}_k$ are f.d. cx v.s.s (83)

If $n \in \mathbb{Z}_{>0}$, $T_n : \left.\begin{matrix} S \\ M \end{matrix}\right\}_k (\Gamma_1(N)) \circlearrowleft$

if $f = \sum_n a_n(f) q^n$, $q = e^{2\pi i z}$ then

$$f | T_n = \sum_{\substack{d | n, \\ (d,N)=1}} d^{k-1} \sum_{\substack{m \geq 0 \\ d | m}} a_{mn/d^2}(f|\langle d \rangle)\, q^m$$

$\left[\Gamma \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \Gamma\right]$ Also: $S_n = n^{k-2}\langle n \rangle$ for $(n,N)=1$

It's not clear that $f|T_n \in \underset{M}{S}_k(\Gamma_1(N))$ for $f \in \underset{M}{S}_k(\Gamma_1(N))$ from this def^n

- look at dbl cosets

Set $\left.\begin{matrix} H_k(\Gamma) \\ h_k(\Gamma) \end{matrix}\right\} = \mathbb{Z}$-alg gen by the T_n in $\text{End}_{\mathbb{C}}\left(\begin{matrix} M_k \\ S_k \end{matrix}\, \Gamma\right)$ $\Gamma = \Gamma_1(N), \Gamma_0(N)$

People usually use $\mathbb{T}$. R.T got Hah from Hida who may well be the only other person who uses it.

If $R \subseteq \mathbb{C}$ is a subring

$\underset{M}{S}_k(\Gamma, R) = \{ f \in \underset{M}{S}_k(\Gamma) \mid a_n(f) \in R\ \forall n > 0 \}$

- note not $n = 0$ even for M

If $\mathbb{Q} \subseteq R$, $M_k(\Gamma, R) = \{ f \in M_k(\Gamma) \mid a_n(f) \in R\ \forall n \geq 0 \}$

Fact: $\left.\begin{matrix} M \\ S \end{matrix}\right\}_k (\Gamma, R)$ is a finite free R-module & the natural map $\underset{S}{M}_k(\Gamma,R) \otimes_R \mathbb{C} \to \underset{S}{M}_k(\Gamma,\mathbb{C})$ ~~is an iso.~~

~~$\left.\begin{matrix} M \\ S \end{matrix}\right\}_k (\Gamma, R)$ is a lattice.~~

$h_k(\Gamma) \times S_k(\Gamma, \mathbb{Z}) \to \mathbb{Z}$, a pairing

$(T, f) \mapsto a_1(f|T)$. This is perfect. (a perfect duality)

ie induces an iso $\mathrm{Hom}(h_k(\Gamma), \mathbb{Z}) \cong S_k(\Gamma, \mathbb{Z})$

& $\mathrm{Hom}(S_k(\Gamma, \mathbb{Z}), \mathbb{Z}) \cong h_k(\Gamma)$.

k>0 r sthg. Similarly H_k, M_k. This is why we have no cond^n on a_0. (84)

We have the Petersson inner product $(\,,): S_k(\Gamma) \times M_k(\Gamma) \to \mathbb{C}$

$$(f,g) = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{h})} \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)}\,(\mathrm{Im}\,z)^k\,d\mu$$

eg $\mu = \frac{dx\,dy}{y^2}$

This is an inner product on $S_k \times S_k$.

T_n is self-adjoint wrt $(\,,)$ if $(n,N)=1$ ($\Gamma = \Gamma_0(N), \Gamma_1(N)$)

Next time: newforms

3/2/92 If $N | M$ & $d | (M/N)$ then $\exists$ map

$[d]: \overset{S}{M}_k(N) \hookrightarrow \overset{S}{M}_k(M)$

$f(z) \mapsto f(dz) = d^{1-k} f|\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$

$[d]$ commutes with T_n if $(n,M)=1$. (85)

Nb eg $d=1$ $\forall N$ gives us overlaps.

Define $S_k(\Gamma_1(M))^{old} = \bigoplus_{\substack{N|M \\ d|M/N}}^{N \neq M} S_k(\Gamma_1(N))\big|[d]$ - stable under $h_k(\Gamma_1(N))$

(86)

& $M_k(\Gamma_1(M))^{old} = \sum_{\substack{N|M \\ d|M/N}}^{N \neq M} M_k(\Gamma_1(N))\big|[d]$; - stable under H_k.

Set $S_k(\Gamma_1(N))^{new}$ = orthog. cplmnt. of $S_k(\Gamma_1(N))^{old}$, preserved by $h_k(\Gamma_1(N))$

$S_k(\Gamma_1(N))^{new}$ is a direct sum of 1-dim^l $h_k(\Gamma_1(N))$-eigensp

An eigenform $f \in S_k(\Gamma_1(N))^{new}$ (of $h_k(\Gamma_1(N))$ with $a_1(f) = 1$ is called a newform.

(Newforms are canonical representatives of eigenforms - $\exists$ too many eigenforms.)

If f & g are eigenforms of the Hecke operators of level N and M resp., we say $f \sim g$ iff $\frac{a_p(f)}{a_1(f)} = \frac{a_p(g)}{a_1(g)}$ for all but finitely many p prime.

(thm) $\Leftrightarrow$ $\underbrace{\frac{a_n(f)}{a_1(f)}}_{\substack{\text{eigen} \\ \text{value of} \\ T_n \text{ on } f}} = \frac{a_n(g)}{a_1(g)}$ $\forall n$ s.t. $(n, NM) = 1$.

($\mathcal{I}$ gives $a_1(f) \neq 0$ as f is an eigenform for all T_p)

Certainly an equiv. rel^n

Fact: $\exists!$ newform f in each $\sim$ class. If $g \sim f$ then the level of f divides the level of g. Conversely, if level of f $| N$ $\exists$ eigenform g of level N s.t. $g \sim f$.

Newforms should play the a role of algebraic Hecke characters in the case $n=2$ of Langlands generalisation of CFT.

NB Knowing the newforms we can work out, bases for $S_k(\Gamma_1(N))$:

$S_k(\Gamma_1(N))$ has a basis $\{ f|[d] : f \text{ is a newform}, (\text{level } f)\times d \mid N \}$

easy pf by ind? certainly this shows they span. (conversely check it now I guess.

Same trick works for M_k except orthog cplmnt of M_k^{old} doesn't make sense as $(,)$ isn't an inner product on M_k

Use Eisenstein series - you can identify which ones you want to be 'old' & which 'new'.

G) Adelic approach

He spent most of the morning trying to get the normalisations right, & even now he's not sure it's OK. He apologises.

Lemma 1) $GL_2(\mathbb{Q}_p) = B(\mathbb{Q}_p)\, GL_2(\mathbb{Z}_p)$, $\quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_2 \right\}$

$SL_2(\mathbb{Q}_p) = B'(\mathbb{Q}_p)\, SL_2(\mathbb{Z}_p)$ $\quad B' = B \cap SL_2$

2) $\mathbb{A}^\times = \mathbb{Q}^\times \hat{\mathbb{Z}}^\times \mathbb{R}^\times_{>0}$

3) $GL_2(\mathbb{A}^\infty) = B(\mathbb{Q})\, GL_2(\hat{\mathbb{Z}})$

$SL_2(\mathbb{A}^\infty) = B'(\mathbb{Q})\, SL_2(\hat{\mathbb{Z}})$

4) $SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$

5) $SL_2(\mathbb{Q})\, SL_2(\mathbb{R}) \subseteq SL_2(\mathbb{A})$ & image is dense.

(Strong approx thm)

1, 4, 5 are very general facts (true in much greater generality)

6) If $U \subset GL_2(\mathbb{A}^\infty)$ is $\overset{\text{subgp}}{\text{open}}$ & if $\det U = \hat{\mathbb{Z}}^\times$ then

$$GL_2(\mathbb{A}) = GL_2(\mathbb{Q})\, U\, \underset{\substack{\uparrow \\ \text{+ve det}^t}}{GL_2^+(\mathbb{R})} \qquad (\mathbb{A}^\infty = \text{"finite adèles"})$$

7) If $U \subseteq GL_2(\mathbb{A}^\infty)$ is $\overset{\text{subgp}}{\text{open}}$ then $\exists\, t_1, \dots, t_r \in \mathbb{A}^\times$ with

$$\mathbb{A}^\times = \coprod_{i=1}^{r} \mathbb{Q}^\times t_i (\det U) \cdot \mathbb{R}^\times_{>0}.$$

Choose $g_i \in GL_2(\mathbb{A})$ with $\det g_i = t_i$. then

$$GL_2(\mathbb{A}) = \coprod_{i=1}^{r} GL_2(\mathbb{Q})\, g_i\, U\, GL_2^+(\mathbb{R})$$

Pf of some of these things

1) The Iwasawa decomposition.

eg SL_2: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}_p)$.

By premultiplying by upper Δlar matrices we want to get it into $\mathbb{Z}_p$, & indeed $SL_2(\mathbb{Z}_p)$.

First choose $\alpha \in \mathbb{Q}_p^\times$ st. αc & $\alpha d \in \mathbb{Z}_p$, & one is a unit.

Premultiply by $\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$: WLOG we can reduce to the case when $c, d \in \mathbb{Z}_p$ & one is a unit.

$\alpha = p^{-\text{smallest power of } p \text{ dividing } c \text{ and } d}$

ie $(c,d) = 1$

Suppose $v_p(c) = 0$. Apply $\begin{pmatrix} 1 & -a/c \\ 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 0 & -1/c \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}_p)$$

The case $v_p(d) = 0$ is exactly similar.

GL ex (88)

2) ex

3) ~~From~~ case GL_2 From 1) we see

$$GL_2(\mathbb{A}^\infty) = B(\mathbb{A}^\infty)\, GL_2(\hat{\mathbb{Z}}) \quad (\text{ex}) \ (89)$$

$\mathbb{A}^\infty$ def

$\mathbb{A}^\infty \subseteq \mathbb{A}$

1)

$$\therefore \text{STP} \quad B(\mathbb{A}^\infty) = B(\cancel{\mathbb{Q}})\, B(\hat{\mathbb{Z}})$$

If $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in B(\mathbb{A}^\infty)$ then by ~~2)~~ $\exists a, c \in \mathbb{Q}^\times$ st $a^{-1}\alpha, c^{-1}\gamma \in \hat{\mathbb{Z}}^\times$

Then it will do to show $\begin{pmatrix} a & \\ & c \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ 0 & \gamma' \end{pmatrix} \in B(\mathbb{Q})B(\hat{\mathbb{Z}})$

$\alpha', \gamma' \in \hat{\mathbb{Z}}^\times$

Choose $u \in \mathbb{Q}$ s.t. $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha' & \beta' \\ 0 & \gamma' \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' - u\gamma' \\ 0 & \gamma' \end{pmatrix} \in B(\hat{\mathbb{Z}})$

ie choose u s.t. $\beta'/\gamma' \in u + \hat{\mathbb{Z}}$. This is OK because $\mathbb{A}^\infty = \mathbb{Q} + \hat{\mathbb{Z}}$ (90)

4) Elementary fiddling about with matrices.

Let $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})$. Choose $a, c \in$ ~~Q~~ $\mathbb{Z}$ st $a \equiv \bar{a}$ (N)

$c \equiv \bar{c}$

$(a, c) = 1$

(OK as $(\bar{a}, \bar{c}) = \mathbb{Z}/N\mathbb{Z}$)

Then $\exists b, d$ st $ad - bc = 1$

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})$

$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$= \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

Now lift this $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $u \in \mathbb{Z}$

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ (N).

So $\twoheadrightarrow$.

5) is SAT for SL_2. Let $U_N' = \{ \gamma \in SL_2(\hat{\mathbb{Z}}) \mid \gamma \equiv 1 \, (N) \}$

These form a basis of open nbds of the identity of $SL_2(\mathbb{A}^\infty)$ (91)

It will do to show $SL_2(\mathbb{Q}) U_N' = SL_2(\mathbb{A}^\infty) \; \forall N.$

$\therefore$ [illegible] but $B(\mathbb{Q}) SL_2(\hat{\mathbb{Z}}) = SL_2(\mathbb{A}^\infty)$

$\therefore$ STP $SL_2(\mathbb{Z}) U_N' = SL_2(\hat{\mathbb{Z}})$ (bring $B_2(\mathbb{Q})$ on left)

But this follows from 4) because $SL_2(\mathbb{Z}) \to SL_2(\hat{\mathbb{Z}}) / U_N' \cong SL_2(\mathbb{Z}/N\mathbb{Z})$ is surj. (92)

'gosh' (after noticing its 1:48)

6) Let $g \in GL_2(\mathbb{A})$. Let $\det g \in \mathbb{A}^\times$, by 2), be $\underset{\mathbb{Q}^\times}{\underset{\cap}{\alpha}} (\det u) \underset{\mathbb{R}^\times_{>0}}{\underset{\cap}{\beta}}$ & $u \in U$

Then look at $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^{-1} g u^{-1} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in SL_2(\mathbb{A})$ (as $\det u =$ [illegible])

$5) \Rightarrow \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^{-1} g u^{-1} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \gamma v \delta,$ $v \in U \cap SL_2(\mathbb{A}^\infty)$
$\delta \in SL_2(\mathbb{R})$ (93)
$\gamma \in SL_2(\mathbb{Q})$

$\therefore g = \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \gamma \right) (vu) \left(\delta \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \right)$. (δ, v, u commute as they're supported at infinite / finite adeles)

7) Similarly (ex) (94)

Strong approximation May [illegible] 98

Say K is a number field, $\mathfrak{p}$ a prime See RLT's 1992 notes p62 or so.

①) Lemma $SL_2(K_\mathfrak{p}) = B'(K_\mathfrak{p})\, SL_2(\mathcal{O}_\mathfrak{p})$ ($\mathcal{O}_\mathfrak{p}$ = integers in $K_\mathfrak{p}$ = completion
$B'(K_\mathfrak{p})$ = upper Δ in $SL_2(K_\mathfrak{p})$

Pf $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ LHS. Choose $\alpha \in K_\mathfrak{p}^\times$ s.t. $\alpha c, \alpha d \in \mathcal{O}_\mathfrak{p}$ & one is a unit. ($B' = \begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}$ in SL_2)

STP $\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B'(K_\mathfrak{p})\, SL_2(\mathcal{O}_\mathfrak{p})$

$\begin{pmatrix} \alpha^{-1}a & \alpha^{-1}b \\ \alpha c & \alpha d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

$c', d' \in \mathcal{O}_\mathfrak{p}$, one a unit.

Case 1) : c' a unit. Premultiply by $\begin{pmatrix} 1 & -a'/c' \\ 0 & 1 \end{pmatrix}$: $\begin{pmatrix} 0 & -1/c' \\ c' & d' \end{pmatrix} \in SL_2(\mathcal{O}_\mathfrak{p})$ ✓

Case 2) d' is a unit. Try $\begin{pmatrix} 1 & -b'/d' \\ 0 & 1 \end{pmatrix}$: $\begin{pmatrix} 1/d' & 0 \\ c' & d' \end{pmatrix} \in SL_2(\mathcal{O}_\mathfrak{p})$ ✓ □

Non-Lemma $SL_2(\mathbb{A}_K^f) = B'(K)\, SL_2(\prod_\mathfrak{p} \mathcal{O}_{K,\mathfrak{p}})$

~~Pf From the previous lemma, $SL_2(\mathbb{A}_K^f) = B'(\mathbb{A}_K^f)\, SL_2(\prod_\mathfrak{p} \mathcal{O}_{K,\mathfrak{p}})$~~

~~so STP $B'(\mathbb{A}_K^f) = B'(K)\, B'(\prod_\mathfrak{p} \mathcal{O}_{K,\mathfrak{p}})$~~

~~But this is false !~~

No – this is false. If we have a non-principal ideal $\mathfrak{p}$ then $\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p}^{-1} \end{pmatrix}$ won't work.

OK,

Lemma Assume K has class no 1. Then $SL_2(\mathbb{A}_K^f) = B'(K)\, SL_2(\prod_\mathfrak{p} \mathcal{O}_{K,\mathfrak{p}})$

Pf From the previous lemma, $SL_2(\mathbb{A}_K^f) = B'(\mathbb{A}_K^f)\, SL_2(\prod_\mathfrak{p} \mathcal{O}_{K,\mathfrak{p}})$

So STP $B'(\mathbb{A}_K^f) = B'(K)\, B'(\prod_\mathfrak{p} \mathcal{O}_{K,\mathfrak{p}})$

– this is false if K doesn't have class no. 1.

Say $B'(\mathbb{A}_K^f) \ni \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$. ~~Write~~ Choose $a \in F^\times$ s.t. $\overset{u}{\overset{\shortparallel}{\alpha/a}} \in \prod \mathcal{O}_\mathfrak{p}^\times$.

$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} u & \beta' \\ 0 & u^{-1} \end{pmatrix}$

Now $\mathbb{A}_K^f = K + \prod \mathcal{O}_\mathfrak{p}$ by SAT so say $\beta' u = k + i$, i integral

Then $\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & \beta' \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} u & \overbrace{\beta' - k/u}^{= i/u} \\ 0 & u^{-1} \end{pmatrix}$ □

$\mathcal{O}=\mathcal{O}_K$ **Lemma** If $\overset{♀}{\mathfrak{n}}\subseteq\mathcal{O}$ & $\bar a,\bar c\in\mathcal{O}/\mathfrak{n}$ generate $\mathcal{O}/\mathfrak{n}$ as an ideal then $\exists$ lifts $a,c\in\mathcal{O}$ s.t. $(a,c)=\mathcal{O}$.

Pf Choose c randomly, $c\neq 0$.

Let $\mathfrak{t}=\prod_{\substack{\mathfrak{p}\text{ s.t.}\\ c\in\mathfrak{p}\ \&\ \mathfrak{p}\nmid\mathfrak{n}}}\mathfrak{p}$

Then $\mathfrak{t}$ & $\mathfrak{n}$ are coprime

& $\mathcal{O}/\mathfrak{t}\mathfrak{n}\to\mathcal{O}/\mathfrak{t}\times\mathcal{O}/\mathfrak{n}$ is an injection, hence a bijection

Choose $a\notin\mathfrak{t}$ s.t. $a\equiv\bar a\ (\mathfrak{n})$

Then $(a,c)=\mathcal{O}$, as if $a,c\in\mathfrak{q}$ we have $a\notin\mathfrak{t}\ \therefore\ \mathfrak{q}\nmid\mathfrak{t}$ & $c\in\mathfrak{q}\ \therefore\ \mathfrak{q}|\mathfrak{n}\ \therefore\ (\mathfrak{n},a,c)\subseteq\mathfrak{q}$ # $\square$

$\mathcal{O}=\mathcal{O}_K$ **Lemma** $SL_2(\mathcal{O})\twoheadrightarrow SL_2(\mathcal{O}/\mathfrak{n}\mathcal{O})$

Pf If $\begin{pmatrix}\bar a&\bar b\\ \bar c&\bar d\end{pmatrix}\in SL_2(\mathcal{O}/\mathfrak{n}\mathcal{O})$ chose a,c coprime lifting $\bar a,\bar c$.

$\exists\ \delta,\beta$ s.t. $a\delta-\beta c=1$.

$$\overline{\begin{pmatrix}a&\beta\\ c&\delta\end{pmatrix}}^{-1}\begin{pmatrix}\bar a&\bar b\\ \bar c&\bar d\end{pmatrix}=\begin{pmatrix}\bar\delta&-\bar\beta\\ -\bar c&\bar a\end{pmatrix}\begin{pmatrix}\bar a&\bar b\\ \bar c&\bar d\end{pmatrix}=\begin{pmatrix}*&*\\ 0&1\end{pmatrix}\in SL_2(\mathcal{O}/\mathfrak{n}\mathcal{O})$$

$$=\begin{pmatrix}1&*\\ 0&1\end{pmatrix}$$

& if we lift this to $\begin{pmatrix}1&u\\ 0&1\end{pmatrix}\in SL_2(\mathcal{O})$

then $\begin{pmatrix}a&\beta\\ c&\delta\end{pmatrix}\begin{pmatrix}1&u\\ 0&1\end{pmatrix}$ works $\square$

Lemma If class no of K is 1 then $SL_2(K)$ is dense in $SL_2(\mathbb{A}^f)$.

Pf Let $U_\mathfrak{n}=\ker\{SL_2(\hat{\mathcal{O}})\to SL_2(\mathcal{O}/\mathfrak{n})\}$

Let's first observe that $B'(K)\,SL_2(\hat{\mathcal{O}})=SL_2(\mathbb{A}^f_K)$ by class no 1, as previous lemma.

Next, $SL_2(\mathcal{O}).U_\mathfrak{n}=SL_2(\hat{\mathcal{O}})$

$\therefore\ SL_2(K).U_\mathfrak{n}=SL_2(\mathbb{A}^f_K)\quad\forall\ \mathfrak{n}\neq 0$.

Say finally $U\subseteq SL_2(\mathbb{A}^f_K)$ is open. Then $U\supseteq\gamma U_\mathfrak{n}$ for some $\mathfrak{n}$, $\gamma\in U$.

$\gamma=gu$, $g\in SL_2(K)$, $u\in U_\mathfrak{n}\ \therefore\ U\supseteq\gamma U_\mathfrak{n}\ni\gamma u^{-1}=g$ $\square$

Cr $SL_2(K).U=SL_2(\mathbb{A}^f_K)$ for all U open, if $cl(K)=1$. Is this necessary?

(NT)

eg $U_N = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv I_2 (N) \}$

$\tilde{U}_N = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} (N) \}$

$U_1(N) = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} (N) \}$

$U_0(N) = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (N) \}$

95 (95) Note $\det \tilde{U}_N = \det U_1(N) = \det U_0(N) = \hat{\mathbb{Z}}^\times$ (75)

$\therefore$ can apply (6)

$U_\infty = \mathbb{R}^\times SO_2(\mathbb{R}) = \{ \lambda \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x^2+y^2=1 \} \subseteq GL_2^+(\mathbb{R})$

$GL_2(\mathbb{R}) / U_\infty \xrightarrow{\sim} \mathfrak{h}^\pm =_{df} \mathbb{C} \setminus \mathbb{R}$ (96)

96 $g \longmapsto gi$

Now let $f \in S_k(\Gamma_1(N))$. Define $\varphi_f : GL_2(\mathbb{A}) \to \mathbb{C}$

via $\gamma u h \longmapsto f(hi) j(h,i)^{-k} (\det h)^{\text{erm, he thinks its 1}}$

note only depends on h.

γ can be [illegible]

$\gamma \in GL_2(\mathbb{Q})$
$u \in U_1(N)$
$h \in GL_2^+(\mathbb{R})$

Check this is well-defined: if $\gamma u h = \gamma' u' h'$
then set $\delta = \gamma'^{-1} \gamma$

$\delta u h = u' h'$ ⇒ look at finite & infinite proj's.

$\delta = u' u^{-1} \in GL_2(\mathbb{Q}) \cap U_1(N) \subseteq GL_2(\mathbb{A}^\infty)$ (projection)

$\delta = h' h^{-1} \in GL_2(\mathbb{Q}) \cap GL_2^+(\mathbb{R}) \subseteq GL_2(\mathbb{R})$ ($proj_{\mathbb{R}}$)

97 $\therefore \det \delta > 0$, $\delta \in GL_2^+(\mathbb{Q}) \cap U_1(N) = \Gamma_1(N)$ (97)

$\therefore f(h'i) j(h',i)^{-k} \det(h') = f(\delta h i) j(\delta h, i)^{-k} \det(h)$

regarding $\Gamma_1(N) \subseteq GL_2^+(\mathbb{R})$

$(f|_k\delta)(z) =$ ~~det~~ $j(\delta,z)^{-k} f(\delta z)$

& $f \in S_k(\Gamma_1(N))$ $\therefore$ $f(\delta h i)\, j(\delta h, i)^{-k} (\det h)$

$= f(hi)\, j(\delta, hi)^k\, j(\delta h, i)^{-k} (\det h)$

$= f(hi)\, j(h,i)^{-k} (\det h)$

So φ_f is well-defined

NB '$\gamma u h = \gamma' u' h'$' is true in $GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R})$ $\therefore$

$\therefore$ $\underset{\text{(in some sense!)}}{\gamma u 1 = \gamma' u' 1}$ & $\gamma 1 h = \gamma' 1 h'$

He wants to record
some properties of φ_f which will eventually characterise it.

$$\varphi_f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / U_1(N) \longrightarrow \mathbb{C}$$

(as φ_f only depends on h)

$\mathbb{R}^\times SO_2(\mathbb{R})$

① $\varphi_f(g u_\infty) = \varphi_f(g)\, j(u_\infty, i)^{-k} (\det u_\infty)$ $\forall u_\infty \in U_\infty \subseteq GL_2^+(\mathbb{R})$

(98)

98 [note $\left(j(h u_\infty, i) = j(h, u_\infty i)\, j(u_\infty, i) \right.$
$\left. j(h, i)\, j(u_\infty, i) \right)$]

② $\forall g \in GL_2(\mathbb{A}^\infty)$, the map $\mathfrak{h} \to \mathbb{C}$

$\{hi,\ h \in GL_2^+(\mathbb{R})\}$

defined by $hi \mapsto \varphi_f(gh)\, j(h,i)^k (\det h)^{-1}$

is well-defined by ① (99) & is holomorphic (for any g)

Pf $g = \gamma u$, $\gamma \in GL_2^+(\mathbb{Q})$, $u \in U_1(N)$. ~~Then $\varphi_f(gh)$~~

$\varphi_f(\gamma_u h) = f(h i)\, j(h,i)^{-k} (\det h)$

Then

$$\varphi_f(gh)\, j(h,i)^k (\det h)^{-1} = \varphi_f(\underset{\substack{\uparrow \\ GL_2(\mathbb{Q}) \\ \cap \\ GL_2(\mathbb{A})}}{\gamma_u} (\underset{\substack{\uparrow \\ GL_2^+(\mathbb{R})}}{\gamma^{-1} h}))\, j(h,i)^k (\det h)^{-1}$$

$$= f(\gamma^{-1} \cancel{h} h i)\, j(\gamma^{-1} h, i)^{-k} \det(\gamma^{-1} h)\, j(h,i)^k (\det h)^{-1}$$

$$= f(\gamma^{-1} h i)\, \cancel{j}(\gamma^{-1}, h i)^{-k} (\det \gamma)^{-1}$$

$$= (f|_k \gamma^{-1})(h i)(\det \gamma)^{k-2}$$

which is holomorphic in hi

$\in S_k(\gamma \Gamma_1(N) \gamma^{-1})$

NO, So later – another def. + proof, ~p70.

③ φ_f is "slowly increasing" ie $|\varphi_f(g)| \leq C \|g\|^N$ $\forall g \in GL_2(\mathbb{A})$, for some csts C, N, indpt of g

Here $\|g\| = \sup\limits_{\substack{v, i, j \\ 1 \leq i,j \leq 2 \\ v \text{ a place}}} \{ |g_{ij}|_v, |(g^{-1})_{ij}|_v \}$ apparently

This may be an exercise, depending on if RI can do it or not. He hasn't done it yet as he spent all morning worrying about his $(\det h)$?

Think this is for all $g \in GL_2(\mathbb{A})$ ↗ ✓

④ $\displaystyle\int_{\mathbb{Q} \backslash \mathbb{A}} \varphi_f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du$ (abelian top gp: ∃ inv't Haar measure (2-sided)) $= 0$ (so it doesn't matter about the normalisation)

$\sim$ cpct (as)

$\mathbb{A} = \mathbb{Q} + \hat{\mathbb{Z}} \times [0,1)$

$\mathbb{A}/\mathbb{Q} \twoheadleftarrow \hat{\mathbb{Z}} \times [0,1)$ ~~[illegible]~~

$\cdots {}_{GL_2(\mathbb{R})}$ [illegible] g

Pf Define $g(\varphi_f)(\_) = \varphi_f(\_\, g)$

$$g(\varphi_f) : {}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A}) / {}_{g U_1(N) g^{-1}} \longrightarrow \mathbb{C} \quad \text{(100)}$$

We'll use the strong approx. thm

Let $g = \gamma u \alpha$, $\gamma \in GL_2(\mathbb{Q})$, $u \in U_1(N)$, $\alpha \in GL_2^+(\mathbb{R})$

Let $h = \varepsilon v \delta$, $\varepsilon \in GL_2(\mathbb{Q})$, $\delta \in GL_2^+(\mathbb{R})$, $v \in g U_1(N) g^{-1}$ $(= \gamma U_1(N) \gamma^{-1})$

(By SAT, $\det g U_1(N) g^{-1} \overset{\hat{\mathbb{Z}}^\times}{=} \ldots$ ok)

Then $g(\varphi_f)(h) = (\text{calc}^n) \overset{\text{(101)}}{=} (f|_k \gamma^{-1})(\delta\gamma\alpha i)\, j(\delta\gamma\alpha, i)^{-k} \det(\delta\gamma\alpha) \times (\det\gamma)^{k-2}$

Rk: in the case $g \in GL_2(\mathbb{A}^\infty)$ then $\gamma = u^{-1} \in GL_2^+(\mathbb{Q}) \subseteq GL_2^+(\mathbb{R})$

Then the formula simplifies to

$$g(\varphi_f)(h) = (f|_k \gamma^{-1})(\delta i)\, j(\delta, i)^{-k} \det(\delta) \det(\gamma)^{k-2}$$

This is by the hyp. (eh?)

$\mathbb{Q}\backslash\mathbb{A} \leftrightarrow N\hat{\mathbb{Z}} \times {}_{N\mathbb{Z}}\backslash\mathbb{R}$ is in fact iso, for any $N \in \mathbb{Z}_{>0}$

Choose N st. $\begin{pmatrix} 1 & N\hat{\mathbb{Z}} \\ 0 & 1 \end{pmatrix} \subseteq g U_1(\overset{\text{another } N}{N}) g^{-1}$

Then we must evaluate $\displaystyle\int_{N\hat{\mathbb{Z}} \times {}_{N\mathbb{Z}}\backslash\mathbb{R}} (g\varphi_f)\left(\begin{pmatrix} 1 & u^\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u_\infty \\ 0 & 1 \end{pmatrix}\right) du^\infty du_\infty$

$\xi = 1 \quad \delta = u_\infty^{-}$

$$= \mathrm{vol}(N\hat{\mathbb{Z}}) \int_{N\mathbb{Z}\backslash\mathbb{R}} (f|\gamma^{-1})(\overset{\left(\begin{smallmatrix}1&u_\infty\\&1\end{smallmatrix}\right)}{u_\infty}\gamma_{\alpha,i})\, \underset{j(\gamma_{\alpha,\iota})^{-k}}{\underset{\shortparallel}{j(\overset{\left(\begin{smallmatrix}1&u_\infty\\&1\end{smallmatrix}\right)}{u_\infty}\gamma_{\alpha,\iota})^{-k}}}\ \overset{\overset{1 \text{ as } u_\infty \text{ is unipotent}}{\shortparallel}}{\det(u_\infty)}\, du_\infty \quad \text{(102)}$$

when he writes u_∞ he means $\begin{pmatrix}1&u_\infty\\0&1\end{pmatrix}$

~~$\int$~~

$$= \mathrm{vol}(N\hat{\mathbb{Z}}) \int_{N\mathbb{Z}\backslash\mathbb{R}} (f|\gamma^{-1})(z+u_\infty)\, du_\infty \quad \text{(103)}$$

$$= \overset{\text{cst}}{\cancel{\mathrm{vol}(N\hat{\mathbb{Z}})}} \int_{\overset{\cancel{N\mathbb{Z}\backslash\mathbb{R}}}{z}}^{z+N} (f|\gamma^{-1})(w)\, dw$$

~~But $\begin{pmatrix}1&N\\0&1\end{pmatrix} \in \gamma$~~

But $f|\gamma^{-1} \in S_k(\gamma\Gamma_1(N)\gamma^{-1})$

$\{g \in \mathrm{GL}_2(\hat{\mathbb{Z}}) / g \equiv \left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right) (N)\}$

$\downarrow$

$\gamma U_1(N)\gamma^{-1}$

$\shortparallel$

& $\gamma\Gamma_1(N)\gamma^{-1} \ni \begin{pmatrix}1&N\\0&1\end{pmatrix}$ by choice of N, as $gU_1(N)g^{-1} \cap \mathrm{GL}_2^+(\mathbb{Q})$
$= \gamma\Gamma_1(N)\gamma^{-1}$
(ex) (104)

$\neq 0$ ∴ integral vanishes
(as its integral across the cusp) (105)

Tomorrow @ 12:30.

Wed. Recall 3): φ_f slowly increasing. He wants to change def. His original def was cribbed of an article by Jacquet & ? in Corvalis. He's not sure if what he said was true. Here's another def. (He ½-convinced himself ∃ ctrexample to what he said on Mon)

~~φ_f~~ Alternative defn of slowly increasing overleaf.

3) φ_f is slowly increasing

ie. $\forall g \in GL_2(\mathbb{A})$ $\exists$ csts C & M (depending on g) s.t.

$$|\varphi_f(gh_\infty)| \le C\|h_\infty\|^M \quad \forall\ h_\infty \in GL_2^+(\mathbb{R})$$

where $\|h_\infty\| = \sup(|h_\infty|, |h_\infty^{-1}|)$ & $|\cdot|$ is a norm on $M_{2\times 2}(\mathbb{R})$.

Pf

Let $g = \gamma u \alpha$ $u \in U_1(N), \gamma \in GL_2(\mathbb{Q}), \alpha \in GL_2^+(\mathbb{R})$

$\alpha h_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$, $u^2+v^2=1$, $\lambda \in \mathbb{R}^\times$, $y \in \mathbb{R}_{>0}$, $x \in \mathbb{R}$

$y = \det(\alpha h_\infty) / c^2+d^2$, $\lambda = \sqrt{c^2+d^2}$

Then $|\varphi_f(gh_\infty)| = |\varphi_f(\alpha h_\infty)| = |f(\alpha h_\infty i)\, j(\alpha h_\infty, i)^{-k} \det(\alpha h_\infty)|$

$= |f(x+iy)\,\lambda^{2-k} y|$

$\le C y^{1-k/2} |\lambda|^{2-k}$ by a well-known estimate on cusp forms

[$f(x+iy)y^{k/2}$ bounded on $\mathfrak{h}$ anyway]

$= C(\det \alpha h_\infty)^{1-k/2}$

$\le C' \|\alpha h_\infty\|^{|k/2-1|}$ (106)

$\le C'' \|h_\infty\|^{|k/2-1|}$

Why what he said last time doesnt appear to be true is that α is related to the infinite cpts & the finite cpts (via γ), need to bound $\|\gamma\|$ or sthg.

$g = \begin{pmatrix} \alpha_S & 0 \\ 0 & 1 \end{pmatrix}$

$(\alpha_S)_p = \begin{cases} p & p \in S \\ 1 & p \notin S \end{cases}$

S gets big, (α_S) doesnt

γ is $\prod_{p\in S} p$ gets big

He guesses the errors in the norm on the adele gp.

Set $S_k = \{ \varphi\ GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) \to \mathbb{C} \mid$

(U indpt of g. U depends on φ)

1) $\varphi(gu) = \varphi(g)$ $\forall g \in GL_2(\mathbb{A})$, $u \in U$, some open cpct in $GL_2(\mathbb{A}^\infty)$

analgous to 1 at ∞ →

2) $\varphi(gu_\infty) = \varphi(g)\, j(u_\infty, i)^{-k} \det(u_\infty)$ $\forall u_\infty \in U_\infty (= \mathbb{R}^* SO_2(\mathbb{R}))$

3) $\forall g \in GL_2(\mathbb{A}^\infty)$ we have a map $\mathfrak{h}^\pm \to \mathbb{C}$ (or just restrict to $\mathfrak{h}$) by if $h \in GL_2(\mathbb{R})$ (or $GL_2^+(\mathbb{R})$),

$hi \longmapsto \varphi(gh)\, j(h,i)^k (\det h)^{-1}$

(well-defined by 2)

(107)

& this must be holomorphic

4) φ is slowly increasing (in the sense he's just described)

5) φ is cuspidal, ie $\displaystyle\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) du = 0$ $\forall g \in GL_2(\mathbb{A})$ $\}$

$GL_2(\mathbb{A}^\infty)$ acts on S_k

(108)

$(g\varphi)(h) = \varphi(hg)$ this preserves properties 1) → 5)

<u>Lemma</u> S_k is an admissible $GL_2(\mathbb{A}^\infty)$ module,

ie $U \subseteq GL_2(\mathbb{A}^\infty)$ open $\Rightarrow S_k^U$ f.d

& $v \in S_k \Rightarrow \mathrm{stab}_{GL_2(\mathbb{A}^\infty)} v$ is open

I think we're pulling U down till its cpct

(109)

Pf 2) is part of the assumptions. So it suffices to check 1).

1): Let $GL_2(\mathbb{A}) = \coprod_{j=1} GL_2(\mathbb{Q}) g_j U GL_2^+(\mathbb{R})$, $g_j \in GL_2(\mathbb{A})$ (see p 62 (#7))

Say $\varphi \in S_k^U$

Define $f_j : \mathfrak{h} \to \mathbb{C}$

$hi \longmapsto \varphi(g_j h)\, j(h,i)^{+k} (\det h)^{-1}$ holo, $h \in GL_2^+(\mathbb{R})$.

Then (exercise) (reverse what he did last time) (110)

$f_j \in S_k(\Gamma_j)$ & $\Gamma_j = g_j U g_j^{-1} \cap GL_2^+(\mathbb{Q})$

& $S_k^U \longrightarrow \bigoplus_{j=1}^{r} S_k(\Gamma_j)$ is iso (111) (exercise) (inj easy, surj just do reverse of last time)

$\varphi \mapsto (f_j)$

- in fact all we need is injectivity.

Hence S_k^U f.d. $\square$

$U, U' \subseteq GL_2(\mathbb{A}^\infty)$ open cpct

$[UgU'] : S_k^{U'} \mapsto S_k^U$

$UgU' = \coprod_i g_i U'$ (112) & $\varphi \mapsto \sum_i g_i(\varphi)$

Now check out rel'ships with classical Hecke operators via iso $S_k^U \cong \bigoplus_{j=1}^{r} S_k(\Gamma_j)$ etc

(13)
$S_k^{\hat{U}_N} \xrightarrow{\sim} S_k(\Gamma_N)$

$S_k^{U_1(N)} \xrightarrow{\sim} S_k(\Gamma_1(N))$ note $r=1$ in all 3 cases, as $\det \hat{U}_N$ etc $= \hat{\mathbb{Z}}^\times$ or whatever

$S_k^{U_0(N)} \xrightarrow{\sim} S_k(\Gamma_0(N))$

Series of rks

1) $U_0(N)/_{U_1(N)} \cong (\mathbb{Z}/_{N\mathbb{Z}})^\times$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a \bmod N$

$U_0(N) \supseteq \Gamma_0(N) \ni \sigma_a^{-1}$ ---- a (eh?)

$\sigma_a \in SL_2(\mathbb{Z})$, $\sigma_a \equiv \begin{pmatrix} * & * \\ 0 & a \end{pmatrix}$ (N)

Note → action is twisted here. think of [illegible]

$\varphi_f \Longleftrightarrow f$ i.e. action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $S_k^{U_1(N)}$ & $S_k(\Gamma_1(N))$ coincide. ⑪④

$$\begin{tikzcd} \varphi_f \arrow[d, "a \in (\mathbb{Z}/N\mathbb{Z})^\times"] \arrow[r, Leftrightarrow] & f \arrow[d, "a \in (\mathbb{Z}/N\mathbb{Z})^\times"] \\ (\sigma_a^{-1})(\varphi_f) \arrow[d, equal] & f|_k \sigma_a \arrow[dl, leftrightarrow] \\ \varphi_{f|_k \sigma_a} & \end{tikzcd}$$

$$S_k^{U_1(N)} = \bigoplus_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} S_k^{U_0(N),\chi} \quad (115)$$

$$\underset{\uparrow \text{ UNTWIST ACTON!}}{S_k^{U_0(N),\chi^{-1}}} \cong S_k(\Gamma_0(N),\chi) \quad (116)$$

2) $z \in (\mathbb{A}^\infty)^\times$ $\quad z = \alpha u \quad \alpha \in \mathbb{Q}^\times_{>0}$, $u \in \hat{\mathbb{Z}}^\times$;

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \sigma_a^{-1} u' \qquad a \equiv u(N)$$
$$u' \in U_1(N).$$

If $\underset{\uparrow \text{ Untwisted action}}{\varphi \in S_k^{U_0(N),\chi}}$ then $\varphi \leftrightarrow f$, & $z(\varphi) \overset{(117)}{\longleftrightarrow} \left(f|_k \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \sigma_a \right) \det \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{2-k}$

$$\overset{(118)}{=} \chi(a)\, \alpha^{k-2} f$$

$$= \chi(a)\, \|z\|^{2-k} f$$

$$\chi(a)\|z\|^{2-k} \varphi \longleftrightarrow$$

$$\|x\| = \prod_v |x_v|_v$$

$\therefore$ action of centre is determined, very simply.

3) Define $S_p = [U_1(N) \begin{pmatrix} \pi_p & 0 \\ 0 & \pi_p \end{pmatrix} U_1(N)]$ $\quad p \nmid N$

$$T_p = [U_1(N) \underset{\uparrow \text{ NO! Try } \begin{pmatrix} \pi_p & 0 \\ 0 & 1 \end{pmatrix} \leftarrow \text{Yes!}}{\begin{pmatrix} 1 & 0 \\ 0 & \pi_p \end{pmatrix}} U_1(N)] \qquad \text{where } \pi_p \in \mathbb{A}^\infty$$

$$(\pi_p)_v = \begin{cases} 1 & v \neq p \\ \underset{\uparrow \text{ or any other uniformiser}}{p} & v = p \end{cases}$$

$$S_p(\varphi_f) = \varphi_{f|_k S_p} \quad (119)$$

$\uparrow$ thus S_p is $[\Gamma_1(N) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N)]$

He wore the
in total Grey
trousers jumper it
gone wanna get

(remember that
times when we spent
all lecture writing
on our notes and
chatting away.

(NT)

Similarly for T_p.

is ~~maybe~~ $\begin{pmatrix}\pi_p & 0\\ 0 & 1\end{pmatrix}$ actually.

- i.e. $T_p(\varphi_g) = \varphi_{g|_k T_p}$

Pf of this: first check $U_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi_p\end{pmatrix}U_1(N) = \coprod_{j=0}^{p-1}\begin{pmatrix}\pi_p & a_j\\ 0 & 1\end{pmatrix}U_1(N) \amalg \underbrace{\begin{pmatrix}1 & 0\\ 0 & \pi_p\end{pmatrix}U_1(N)}_{\text{if } p\nmid N}$ (120)

where $a_j \in \mathbb{A}^\infty$, $(a_j)_v \overset{i.e.}{=} \begin{cases} 1 & v \neq p\\ j & v = p\end{cases}$

$$T_p(\varphi_g) = \sum_j \begin{pmatrix}\pi_p & a_j\\ 0 & 1\end{pmatrix}\varphi_g + \underbrace{\begin{pmatrix}1 & 0\\ 0 & \pi_p\end{pmatrix}\varphi_g}_{p\nmid N}$$

He's gonna get it sorted for next time.

Then on S_k we define

$$(\varphi_1, \varphi_2) = \int_{\mathbb{R}^\times_{>0} GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})} \varphi_1(g)\,\varphi_2(g)\,|\det g_\infty|^{k-2}\,dg$$

this exists but it's not cpt w/ sty. $\Gamma\backslash\mathfrak{h}$ not cpt. But finite volume wrt Haar measure exists.

This respects the action of $GL_2($~~erm~~ $\mathbb{A}^\infty)$ nicely, i.e.

$$(g(\varphi_1), g(\varphi_2)) = (\varphi_1, \varphi_2), \quad g \in GL_2(\mathbb{A}^\infty) \qquad (121)$$

( ) is an inner product

S_k is ∞-dim' (and has a ctble basis)

S_k is not complete wrt ( ) ie not a Hilbert space so its not immediate that S_k is a direct sum of irreducibles. Because of admissibility this will be true though.

<u>Lemma</u> $S_k = \bigoplus \pi_i$, π_i irred admiss rep of $GL_2(\mathbb{A}^\infty)$

<u>Sketch pf</u> Choose $U \subseteq GL_2(\mathbb{A}^\infty)$ open cpct

$$S_k = \bigoplus_{\substack{\rho \text{ irred} \\ \text{f.d. reps (of } U\text{, I guess)} \\ \text{(cts)(wrt disc. top on } \mathbb{C})}} S_k^{\rho} \qquad S_k^{\rho} = \sum_{\theta \in \mathrm{Hom}(\rho, S_k)} \mathrm{im}\,\theta$$

Dim $S_k^{\rho} < \infty$ because $S_k^{\rho} \subseteq S_k^{\ker\rho}$, $\ker\rho$ open

RT feels this should be an orthogonal direct sum

If $v_1 \in S_k^{\rho_1}$, $v_2 \in S_k^{\rho_2}$, $(v_1, v_2) \neq 0$ then

$\check{\rho}_1$ contragredient, $\check{\rho}_1$ (cx conj) $\check{\rho}_1 \cong \rho_2$ $(\cong \rho_1)$

- standard argument, as you get a nonzero HM from $S_k^{\rho_1}$ to $S_k^{\bar{\rho}_1}$ $f^{(v_1 = v_2)}$

In ptic, $\bar{\check{\rho}}_1 \cong \rho_1$ hence we have an orthogonal direct sum

So $\bigoplus S_k^{\rho}$ is orthogonal.

So if $W \subseteq S_k$ is an admissible submodule,

$$W = \bigoplus_{\rho} W^{\rho}. \quad \text{Set } W^{\perp} = \bigoplus_{\rho} (W^{\rho})^{\perp \, (\text{in } S_k^{\rho})}$$

$$W \oplus W^{\perp} = S_k$$

Index the ρs: $\rho_1, \rho_2, \ldots$ ($\exists$ only ctbly many of $U = GL_2(\hat{\mathbb{Z}})$ cont by U_n they vanish on, & U/U_n finite)

Take π_1 to be a min^l submodule st $\pi_1^{\rho_1} \neq (0)$.

Then $\pi_1 \neq (0)$, π_1 irred. Then $S_k = \pi_1 \oplus W_1$ etc. $\square$

Now let π_p be an irred admiss rep^n of $GL_2(\mathbb{Q}_p)$ for all p

Assume π_p is unramified for almost all p, say for $p \notin S_0$, S_0 finite

Choose $v_p \in \pi_p^{GL_2(\mathbb{Z}_p)}$, $v_p \neq 0$ for these p.

Define $\bigotimes_p \pi_p = \varinjlim_{\substack{S \text{ finite} \\ S \supseteq S_0}} \bigotimes_{p \in S} \pi_p$

If $S_0 \subseteq S_1 \subseteq S_2$ define $\bigotimes_{p \in S_1} \pi_p \to \bigotimes_{p \in S_2} \pi_p$

$$x \mapsto x \otimes \underset{p \in S_2 \setminus S_1}{v_p}$$

$\uparrow$

he seems to have written x^-. ?

Then $\bigotimes_p \pi_p$ is spanned by elts of the form $x_{p_1} \otimes \cdots \otimes x_{p_r} \otimes v_{p_{r+1}} \otimes v_{p_{r+2}} \otimes \cdots$

Ex 1) $\bigotimes_p \pi_p$ is an admissible rep^n of $GL_2(\mathbb{A}^\infty)$

2) Indpt of choice of v_p

Lemma 1) $\bigotimes_p \pi_p$ is irred.

2) If π is an irred adm rep of $GL_2(\mathbb{A}^\infty)$ then $\pi \cong \bigotimes_p \pi_p$ for some such π_p.

He'll sketch a pf of this

Thm $S_k = \bigoplus \pi_i$ infinite algebraic direct sum, $\pi_i = \bigotimes_p \pi_{i,p}$

If $\pi_{i,p} \cong \pi_{j,p}$ for all but finitely many p then $i=j$

He'll certainly not prove this

Exercise Derive the theory of newforms from this thm & the adelic approach.

In the 12 mins left he'll give a sketch of 1) & the perhaps more surprising 2) of the lemma.

$G = GL_2(\mathbb{Q}_p)$ or $GL_2(\mathbb{A}^\infty)$, $U \subseteq G$ open cpct subgp

$\mathbb{C}[U \backslash G / U]$ is an algebra

π is admissible
π^U is a f.d. $\mathbb{C}[U \backslash G / U]$-module.

Ex π irred $\Leftrightarrow$ $\pi^U / \mathbb{C}[U \backslash G / U]$ irred $\forall U$.

If $U = \prod U_p \subseteq GL_2(\mathbb{A}^\infty)$, $U_p \subseteq GL_2(\mathbb{Q}_p)$ open cpct,
$U_p = GL_2(\mathbb{Z}_p)$ for almost all p

$$\mathbb{C}[U \backslash GL_2(\mathbb{A}^\infty) / U] \cong \bigotimes_p \mathbb{C}[U_p \backslash GL_2(\mathbb{Q}_p) / U_p]$$

restricted w.r.t. $U_p \in \mathbb{C}[U_p \backslash GL_2(\mathbb{Q}_p) / U_p]$
ie put these in when going from a small tensor product to a large one.

1) now check - we can weaken Ex above to π irred $\Leftrightarrow$ $\pi^U / \mathbb{C}[U \backslash G / U]$ irred for some cofinal system of U.

So $(\otimes \pi_p)^U = \otimes \pi_p^{U_p}$ (=1 almost everywhere)

& a finite tensor product of algebras irred rep is irred ∴

$\otimes \pi_p^{U_p}$ is irred / $\bigotimes_p \mathbb{C}[U_p \backslash GL_2(\mathbb{Q}_p) / U_p]$

$\Rightarrow \bigotimes_p \pi_p$ is irred.

— This does 1)

2) π^U is an irred $\otimes_p \mathbb{C}[\underbrace{U_p \backslash GL_2(\mathbb{Q}_p)/U_p}]$-module.

abelian if $U_p = GL_2(\mathbb{Z}_p)$, generated essentially by T_p & S_p & S_p^{-1} or sthg

ie ab for almost all p.

for almost all p, $\mathbb{C}[U_p \backslash GL_2(\mathbb{Q}_p)/U_p]$ acts by a character
ie $p \notin S$, S finite

$$\lambda_p : \mathbb{C}[U_p \backslash GL_2(\mathbb{Q}_p)/U_p] \to \mathbb{C}$$

π^U is irred $\bigotimes_{p\in S} \mathbb{C}[U_p \backslash GL_2(\mathbb{Q}_p)/U_p]$-module

$$\therefore \pi^U \cong \bigotimes_{p\in S} \pi_p^U, \quad \pi^U = \bigotimes_{p\in S} \pi_p^U \otimes \bigotimes_{p\notin S} \lambda_p.$$

If $U \subseteq U'$ $\exists\ \pi_p^{U'} \hookrightarrow \pi_p^U$

Set $\pi_p = \varinjlim_{U} \pi_p^U$

Check π_p admiss. $\bigotimes \pi_p = \pi$.

This is just a sketch.
It wouldn't be difficult to fill in the gaps.

The gap here is due to Jesus Christ
& in particular the Camden Choir!

$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \;\middle|\; c \equiv 0,\ d \equiv 1\ (N) \right\}$

Between (78) & (79).

Kevins.

$f \in S_k(\Gamma_1(N))$

$\phi_f : GL_2(\mathbb{A}) \longrightarrow \mathbb{C}$

$\gamma u g \longmapsto f(gi)\, j(g,i)^{-k} \det g$

$\gamma \in GL_2(\mathbb{Q})$

$u \in U_1(N)$

$g \in GL_2^+(\mathbb{R})$

well-defined: $\gamma u g = \gamma' u' g'$, $\delta =_{df} \gamma'^{-1}\gamma$

$\delta = u' u^{-1}$ so $\delta \in GL_2(\mathbb{Q}) \cap U_1(N) \subset GL_2(\mathbb{A}^\infty)$

$\delta = g' g^{-1}$ so $\delta \in GL_2^+(\mathbb{Q}) \subset GL_2(\mathbb{A}_\infty)$

$\therefore \delta \in GL_2^+(\mathbb{Q}) \cap U_1(N) = \Gamma_1(N)$

$f(g'i)\, j(g',i)^{-k} \det g' = f(gi)\, j(\delta, gi)^{k} (\det \delta)^{-1}\, j(\delta g, i)^{-k} (\det g\delta)$

$= f(gi)\, j(g,i)^{-k} (\det g)$

$\phi_f : {}^{GL_2(\mathbb{Q})}\backslash{}_{GL_2(\mathbb{A})} \longrightarrow \mathbb{C}$ st

1) $\phi_f(gu) = \phi_f(g) \quad \forall u \in U_1(N)$

2) $\phi_f(g u_\infty) = \phi_f(g)\, j(u_\infty, i)^{-k} \det u_\infty \quad \forall u_\infty \in U_\infty$

3) $\forall g \in GL_2(\mathbb{A}^\infty)$ the map

$\mathfrak{h} \longrightarrow \mathbb{C}$

$hi \longmapsto \phi_f(gh)\, j(h,i)^{k} (\det h)^{-1}$

$h \in GL_2^+(\mathbb{R})$

is well defined + holomorphic

4) ϕ_f is slowly increasing,

i.e. $\forall g \in GL_2(\mathbb{A}^\infty)$ $\exists C, M > 0$ st

$|\phi_f(g h_\infty)| \leq C \|h_\infty\|^M \quad \forall h_\infty \in GL_2^+(\mathbb{R})$

where $\|h_\infty\| = \max(|h_\infty|, |h_\infty^{-1}|)$ where $|.|$ any norm on $M_{2\times 2}(\mathbb{R})$.

ϕ_f is cuspidal: i.e.

5) $\int_{\mathbb{Q}\backslash\mathbb{A}} \phi_f\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) du = 0 \quad \forall g \in Gl_2(\mathbb{A})$

Pf $\phi_f(\gamma u h g)$

$\gamma \in Gl_2(\mathbb{Q})$
$h \in Gl_2^+(\mathbb{R})$
$u \in$ ~~[illegible]~~ $gU_1(N)g^{-1}$

$g \in Gl_2(\mathbb{A})$
$g = \alpha v k$
$\alpha \in Gl_2^+(\mathbb{Q})$
$v \in U_1(N)$
$k \in Gl_2^+(\mathbb{R})$

$= \phi_f(\gamma\alpha\,(v\,g^{-1}ug)\;kg^{-1}hg)$

$= \phi_f(\alpha^{-1}h\alpha k) = f(\alpha^{-1}h\alpha k i)\, j(\alpha^{-1}h\alpha k, i)^{-k} \;\det(hk)$

$= (f|_k\alpha^{-1})(h\alpha k i)\;\; j(\alpha^{-1}, i)^{-k}\; (\det\alpha)^{k-1}\;\; j(\alpha^{-1}h\alpha k, i)^{-k} \det(hk)$

if $\alpha \in Gl_2^+(\mathbb{Q})$

$= (f|_k\alpha^{-1})(h\alpha k i)\;\; j(h\alpha k, i)^{-k}\; \det(h\alpha k)\; (\det\alpha)^{k-2}$

If $g \in Gl_2(\mathbb{A}^\infty)$ then we get $(f|_k\alpha^{-1})(hi)\; j(h,i)^{-k}\; \det(h)\; (\det\alpha)^{k-2}$.

Now 3) $\phi_f(gh)\, j(h,i)^k (\det h)^{-1} = (\det\alpha)^{k-2} (f|_k\alpha^{-1})(z)$ $\quad z = hi$

is holo

~~$|\phi_f(gh_\infty)| = |(f|_k\alpha^{-1})(h_\infty i)\; j(h_\infty, i)^{-k} \det(h_\infty)|$~~

[illegible]

$= |\phi_f(g(h_\infty\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}))|$

4) $|\phi_f(gh_\infty)|$ [illegible]

Thus we can assume $h_\infty \in Gl_2^+(\mathbb{R})$. Then

$|\phi_f(gh_\infty)| = |\det\alpha|^{k-2}\, |(f|_k\alpha^{-1})(h_\infty i)\;\; j(h_\infty, i)^{-k} \det h_\infty|$

$$h_\infty = z\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}u & v\\ -v & u\end{pmatrix} \qquad u^2+v^2=1 \qquad \det h_\infty = z^2 y$$

$$|j(h_\infty,i)| = |z| = \left|\frac{\det h_\infty}{\mathrm{Im}\, h_\infty i}\right|^{1/2}$$

$$\therefore |\phi_f(gh_\infty)| = |\det\alpha|^{k-2}\ \left|(f|_k\alpha^{-1})(h_\infty i)\ (\mathrm{Im}\, h_\infty i)^{k/2}\ (\det h_\infty)^{1-k/2}\right|$$

$$\leq C\ \sup\{(\mathrm{Im}\, h_\infty i), (\mathrm{Im}\, h_\infty i)^{-1}\}^{k/2}(\det h_\infty)^{1-k/2}$$

$$\leq C'\ \|h_\infty\|^{2k}$$

⑤ $\phi_f\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}g\right) = (f|_k\alpha^{-1})(\alpha k i + u_\infty)\ j(\alpha k, i)^{-k}\det(\alpha k)(\det\alpha)^{k-2}$

if $u \in N\hat{\mathbb{Z}}\times\mathbb{R}$ with N chosen st $\begin{pmatrix}1 & N\hat{\mathbb{Z}}\\ 0 & 1\end{pmatrix} \subset gU_1(N)g^{-1}$

$$\therefore \int_{\mathbb{Q}\backslash\mathbb{A}} \phi_f\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}g\right)du = \mathrm{vol}(N\hat{\mathbb{Z}})\ j(\alpha k, i)^{-k}\det(\alpha k)(\det\alpha)^{k-2}\int_{\mathbb{R}/N\mathbb{Z}}(f|_k\alpha^{-1})(\alpha k i + u_\infty)\,du_\infty$$

$$= 0 \quad \text{because } \begin{pmatrix}1 & N\\ 0 & 1\end{pmatrix} \in gU_1(N)g^{-1}\cap GL_2^+(\mathbb{Q})$$
$$= \alpha\Gamma_1(N)\alpha^{-1}$$

<u>Exercise</u> $f \longmapsto \phi_f$ gives a bijection between $S_k(\Gamma_1(N))$ and the functions on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$ satisfying 1) — 5)

<u>Exercise</u> $f \longmapsto \phi_f$ gives a bijection between $M_k(\Gamma_1(N))$ and the functions on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$ satisfying 1) — 4).

$M_k = \Big\{ \phi : GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) \longrightarrow \mathbb{C} \;\Big|$ 1) $\phi(gu) = \phi(g) \quad \forall u \in U$

$U \subset GL_2(\mathbb{A}^\infty)$ open depending on ϕ but not g.

2) $\phi(gu_\infty) = j(u_\infty, i)^{-k} (\det u_\infty)\ \phi(g) \quad \forall u_\infty \in U_\infty$

3) $\forall g \in GL_2(\mathbb{A}^\infty)$ the map

$$\begin{aligned} \mathfrak{h}^\pm &\longrightarrow \mathbb{C} \\ hi &\longmapsto \phi(gh)\, j(h,i)^k (\det h)^{-1} \end{aligned}$$

is well defined by 2) and holomorphic.

4) ϕ is slowly increasing $\Big\}$

$S_k = \Big\{ \phi \in M_k \;\Big|\; \forall g \in GL_2(\mathbb{A}) \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0 \Big\}$

$GL_2(\mathbb{A}^\infty)$ acts on M_k and S_k by $g(\phi)(-) = \phi(-g)$.

__lemma__ If U ($\subset GL_2(\mathbb{A}^\infty)$) is open compact then $GL_2(\mathbb{A}) = \coprod_{j=1}^{r} GL_2(\mathbb{Q})\, g_j U\, GL_2^+(\mathbb{R})$, and

$M_k^U \cong \bigoplus_{j=1}^{r} M_k(\Gamma_j)$ where $\Gamma_j = g_j U g_j^{-1} \cap GL_2^+(\mathbb{Q})$ and the map is

$\phi \longmapsto (f_j) \qquad f_j(hi) = \phi(g_j h)\, j(h,i)^k (\det h)^{-1}.$

Similarly $S_k^U \cong \bigoplus_{j=1}^{r} S_k(\Gamma_j)$.

Pf Exercise.

~~Or~~ S_k and M_k are admissible $Gl_2(\mathbb{A}^\infty)$ - modules, i.e.

1) the fixed points of an open subgroup are finite dimensional

2) stabilisers of single vectors are open.

If $U, U' \subset Gl_2(\mathbb{A}^\infty)$ are open compact then $UgU' = \coprod_{i=1}^{r} g_i U'$ a finite disjoint union. Define a linear map (Hecke operator):

$$[UgU'] : S_k^{U'} \longrightarrow S_k^{U}$$
$$\phi \longmapsto \sum g_i(\phi).$$

(Exercise: This is well defined.)

$$U_0(N)/U_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d$$

If $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ pick $\sigma_d \in$ [illegible] $\Gamma_0(N)$, $\sigma_d \equiv \begin{pmatrix} * & * \\ 0 & d \end{pmatrix} (N)$.

$(\mathbb{Z}/N\mathbb{Z})^\times$ acts on $S_k^{U_1(N)}$ and we get $S_k^{U_1(N)} = \underbrace{\bigoplus}_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} S_k^{U_0(N), \chi}$.

Suppose $\phi \in S_k^{U_1(N)}$ [illegible] $f \in S_k(\Gamma_1(N))$].

Then $\phi \in S_k^{U_1(N)}$ [illegible] $\chi(d)\phi \quad \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times$

[illegible] $f|\sigma_d^{-1} = \chi(d) f \quad \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times$

$\Leftrightarrow f \in S_k(\Gamma_0(N), \chi^{-1})$.

Also let $z \in (\mathbb{A}^\infty)^\times$ then $z = \alpha \sigma_d u$ where $\alpha \in \mathbb{Q}_{>0}^\times$, $d \equiv \alpha^{-1} z \ (N)$, $u \in U_1(N)$

and so $z(\phi) = \sigma_d(\phi(\_\ \alpha^{-1})) \qquad \alpha^{-1} \in \mathbb{R}_{>0}^\times$

$$= \alpha^{k-2} \chi(d)\, \phi = \|z\|^{2-k} \tilde{\chi}(z)\, \phi$$

where $\tilde{\chi}$ is the grossencharacter associated to χ by $\mathbb{A}^\times / \mathbb{Q}^\times \mathbb{R}_{>0}^\times \cong \hat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$

(Note that we must have $\tilde{\chi}(-1) = (-1)^k$.)

Let $S_p = \left[U_1(N)\begin{pmatrix}\pi_p & 0\\ 0 & \pi_p\end{pmatrix}U_1(N)\right]$ if $p \nmid N$

$T_p = \left[U_1(N)\begin{pmatrix}\pi_p & 0\\ 0 & 1\end{pmatrix}U_1(N)\right]$

where $(\pi_p)_v = \begin{cases} 1 & v \neq p \\ p & v = p \end{cases}$

$$S_p(\phi_f) = p^{k-2}\,\sigma_p^{-1}(\phi_f) = \phi_{(p^{k-2} f|_k \sigma_p)} = \phi_{(f|S_p)}$$

$$T_p(\phi_f) = \sum_{j=0}^{p-1}\begin{pmatrix}\pi_p & \alpha_j\\ 0 & 1\end{pmatrix}(\phi_f) + \underbrace{\left[\begin{pmatrix}1 & 0\\ 0 & \pi_p\end{pmatrix}(\phi_f)\right]}_{\text{if } p \nmid N} \qquad (\alpha_j)_v = \begin{cases} 0 & v \neq p \\ j & v = p. \end{cases}$$

$$= \left(\sum_{j=0}^{p-1}\phi_{f|_k\begin{pmatrix}p & j\\ 0 & 1\end{pmatrix}^{-1}} + \underbrace{\phi_{f|_k\left(\sigma_p^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\right)^{-1}}}_{\text{if } p \nmid N}\right)p^{k-2}$$

$$= \left(\sum_{j=0}^{p-1}\phi_{f|_k\begin{pmatrix}1 & -j\\ 0 & p\end{pmatrix}} + \underbrace{\phi_{f|_k\sigma_p\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}}}_{\text{if } p \nmid N}\right) \quad \text{as } f|_k\begin{pmatrix}p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix} = p^{2-k} f$$

$= \phi_{(f|}$ [illegible]

Define $(\phi_1, \phi_2) = \int_{\mathbb{R}^{\times}_{>0} G(\mathbb{Q})\backslash G(\mathbb{A})} \phi_1(g)\,\overline{\phi_2(g)}\,|\det g_\infty|^{k-2}\,dg$ for $\phi_1, \phi_2 \in M_k$, one in S_k.

Here dg is any fixed Haar measure.

Exercise $(\,,\,)$ is well defined.

If $g \in Gl_2(\mathbb{A}^\infty)$ we have $(g(\phi_1), g(\phi_2)) = (\phi_1, \phi_2)$.

<u>Lemma</u> 1) We can write $M_k = S_k \oplus G_k$ as an orthogonal direct sum of admissible $GL_2(\mathbb{A}^\infty)$-modules.

2) $S_k = \oplus \pi_i$ where π_i are irreducible admissible $Gl_2(\mathbb{A}^\infty)$-modules and the direct sum is orthogonal.

<u>Pf</u> (sketch) Fix $U \subset Gl_2(\mathbb{A}^\infty)$ open compact.

We can write $S_k = \oplus S_k^\rho$

$M_k = \oplus M_k^\rho$ (w.r.t. discrete top on G)

where ρ runs over finite dimensional cts/ irreducible representations of U and $S_k^\rho = \sum_{\phi \in Hom(\rho, S_k)} Im\, \phi$.

<u>Exercises</u> 1) $\dim M_k^\rho < \infty$

2) If $S_k^\rho \neq (0)$ then $\rho \cong \overline{\check{\rho}}$

3) [illegible] $\bigoplus_\rho S_k^\rho$ is an orthogonal direct sum and if $\rho \not\cong \rho'$ then S_k^ρ and $M_k^{\rho'}$ are orthogonal

4) If $W \subset S_k$ is an admissible $Gl_2(\mathbb{A}^\infty)$ submodule then [illegible] $M_k = W \oplus W^\perp$ and $W^\perp$ is an admissible $Gl_2(\mathbb{A}^\infty)$-module

5) There are only countably many possible ρ

Number the $\rho_1, \rho_2, \ldots$ Define π_i inductively such that $S_k = \pi_1 \oplus \cdots \oplus \pi_r \oplus W_r$, as follows. Choose n_r minimal st $W_r^{\rho_{n_r}} \neq (0)$ and choose π_{r+1} to be a minimal subrepresentation of W_r with $\pi_{r+1}^{\rho_{n_r}} \neq (0)$. Then each π_i is irred. Moreover given n $\exists R$ st $W_R^{\rho_n} = (0)$. Thus $\bigcap_r W_r = (0)$ and $S_k = \oplus \pi_i$.

If I is any set of indices, V_i is a vector space for $i \in I$ and $v_i \in V_i$ is a distinguished vector for all but finitely many i. We define a restricted tensor product:

$$\widehat{\bigotimes_I} V_i = \varinjlim_{\substack{S_0 \subseteq S \subset I \\ S \text{ finite}}} \bigotimes_S V_i$$

where S_0 is the set of i s.t. V_i does not have a distinguished vector and if $S_0 \subset S_1 \subset S_2 \subset I$ then

$$\bigotimes_{S_1} V_i \longrightarrow \bigotimes_{S_2} V_i$$

$$x \longmapsto x \otimes \bigotimes_{S_2 - S_1} v_i$$

$\widehat{\bigotimes_I} V_i$ is spanned by $\{ \bigotimes_I x_i \mid x_i = v_i$ for all but finitely many $i \}$.

eg. 1) If $U = \prod_p U_p$, $U_p \subset Gl_2(\mathbb{Q}_p)$ open compact
$U_p = Gl_2(\mathbb{Z}_p)$ for all but finitely many p

then

$$\mathbb{C}[U \backslash Gl_2(\mathbb{A}^\infty) / U] \cong \widehat{\bigotimes_p} \mathbb{C}[U_p \backslash Gl_2(\mathbb{Q}_p) / U_p]$$

where the product is restricted w.r.t. $U_p \in \mathbb{C}[U_p \backslash Gl_2(\mathbb{Q}_p)/U_p]$. This preserves the algebra structure.

2) If π_p is an admissible irreducible representation of $Gl_2(\mathbb{Q}_p)$ ~~for~~ such that for all but finitely many p π_p is unramified, then $\widehat{\bigotimes_p} \pi_p$ is an admissible $Gl_2(\mathbb{A}^\infty)$-module. The product is restricted w.r.t. any $v_p \in \pi_p^{Gl_2(\mathbb{Z}_p)}$ $(v_p \neq 0)$.
Up to isomorphism $\widehat{\bigotimes_p} \pi_p$ does not depend on the choice of v_p.

$$\left(\widehat{\otimes}_p \pi_p\right)^U = \widehat{\otimes}_p \left(\pi_p^{U_p}\right) \quad \text{if } U = \prod_p U_p,\ U_p = Gl_2(\mathbb{Z}_p) \text{ for all but finitely many } p.$$

$(\dim \pi_p^{U_p} = 1$ for all but finitely many $p)$.

Rk If $G = GL_2(\mathbb{Q}_p)$ or $GL_2(\mathbb{A}^\infty)$ and π is an admissible rep of G then

π^U irred / $\mathbb{C}[U\backslash G/U]$ for arbitrarily small U $\Rightarrow$ π irred
$\Rightarrow$ π^U irreducible for all U over $\mathbb{C}[U\backslash G/U]$

[if $A \subset \pi^U$ is a proper $\mathbb{C}[U\backslash G/U]$-submodule then $\tilde{A}$ the G-module generated by A is a proper submodule, in fact $\tilde{A} \cap \pi^U = A$, because if $\tilde{\alpha} = \sum g_i a_i$, $a_i \in A$, $\tilde{\alpha} \in \pi^U$ and if we set $U' = (\bigcap_i g_i U g_i^{-1}) \cap U$ we have that $\tilde{\alpha}[U:U'] = \sum_i [UU'][U'g_iU]a_i \in A$.]

Thus $\widehat{\bigotimes}_p \pi_p$ is irreducible as $\widehat{\bigotimes}_p \pi_p^{U_p}$ / $\widehat{\bigotimes}_p \mathbb{C}[U_p\backslash GL_2(\mathbb{Q}_p)/U_p]$ is irreducible.

Lemma If π is an admiss irred rep of $GL_2(\mathbb{A}^\infty)$ then $\pi \cong \widehat{\bigotimes}_p \pi_p$ for certain irred admiss reps π_p of $GL_2(\mathbb{Q}_p)$, which are uniquely determined by π.

(SKETCH)
Uniqueness follows because $\pi \cong \widehat{\bigotimes}_p \pi_p \Rightarrow \left(\mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(\rho, \pi) \neq (0)\right.$ for ρ an irred admiss rep of $GL_2(\mathbb{Q}_p)$ $\left.\Leftrightarrow \rho \cong \pi_p.\right)$

For existence $\pi^{\prod U_p}$ is an irreducible $\widehat{\bigotimes}_p \mathbb{C}[U_p\backslash GL_2(\mathbb{Q}_p)/U_p]$-module. Moreover for $p \notin S$ for some finite set S, $\mathbb{C}[U_p\backslash GL_2(\mathbb{Q}_p)/U_p]$ is abelian and so acts by scalars on $\pi^{\prod U_p}$. Thus $\pi^{\prod U_p}$ is an irreducible $\bigotimes_{p\in S} \mathbb{C}[U_p\backslash GL_2(\mathbb{Q}_p)/U_p]$ module, which implies $\pi^{\prod U_p} \cong \bigotimes_{p\in S} \pi_p^{\prod U_p}$

$$\pi = \varinjlim \pi^{\prod_p U_p} = \varinjlim \bigotimes_{p\in S} \pi_p^{\prod_p U_p} = \widehat{\bigotimes_p} \pi_p \text{ for some } \pi_p.$$

Check out the handout. Resumé of last lecture.

We'll define $U_1(N) = \{ \gamma \in GL_2(\hat{\mathbb{Z}}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \ (N) \}$

$$\tilde{U}_N = \{ \gamma \in GL_2(\hat{\mathbb{Z}}) \mid \gamma \equiv \left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right) \ (N) \}$$

Then $S_k^{U_1(N)} \xrightarrow{\sim} S_k(\Gamma_1(N))$ & this iso preserves the action of T_p & S_p.

$U_0(N) / U_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod n$

$$S_k^{U_1(N)} = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} S_k^{U_0(N), \chi}, \quad S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(\Gamma_0(N), \chi)$$

$\uparrow$
($U_0(N)$ acts, & U_1 acts trivially)

$S_k^{U_0(N), \chi} \cong S_k(\Gamma_0(N), \chi^{-1})$ (not what he said last time)

$T_p = [U_1(N) \left(\begin{smallmatrix} \varpi_p & 0 \\ 0 & 1 \end{smallmatrix}\right) U_1(N)]$, $S_p = [U_1(N) \left(\begin{smallmatrix} \varpi_p & 0 \\ 0 & \varpi_p \end{smallmatrix}\right) U_1(N)]$

$$(\varpi_p)_v = \begin{cases} 1 & p \neq v \\ p & p = v \end{cases}$$

$UgU' : S_k^{U} \to S_k^{U'}$, & if $UgU' = \coprod_{i=1}^{r} g_i U'$, it's $\varphi \mapsto \sum_{i=1}^{r} g_i(\varphi)$

Finally, $S_k = \bigoplus_{i \in I} \pi_i$, π_i irred admiss rep of $GL_2(\mathbb{A}^\infty)$

$\pi_i = \hat{\bigotimes_p} \pi_{i,p}$, $\pi_{i,p}$ irred admiss rep of $GL_2(\mathbb{Q}_p)$

Thm (strong multiplicity one) If for all but finitely many p, $\pi_{i,p} \cong \pi_{j,p}$, then $i = j$. In ptic $i \neq j \Rightarrow \pi_i \ncong \pi_j$.

Recall the exercise: develop the theory of newforms from an adelic pt of view, based on stuff he's proved / told us about.

Also, we can do all this with

$$M_k = \left\{ \text{———} \;\middle|\; \begin{matrix} 1) \\ 2) \\ 3) \\ 4) \end{matrix} \quad \text{but leave out 5)} \leftarrow \text{'cuspidal'} \right\}$$

$$\underset{\text{Hecke matches up.}}{M_k^{U_1(N)} \cong M_k(\Gamma_1(M))}, \quad M_k = S_k \oplus G_k \text{ as admissible } GL_2(\mathbb{A}^\infty)\text{-modules}$$

G_k is even semisimple he thinks in this case

Now we'll do some automorphic forms

$\tilde{U}_\infty = \langle U_\infty, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle = \mathbb{R}^\times O_2(\mathbb{R}) \subset GL_2(\mathbb{R})$

An admissible $GL_2(\mathbb{R})$-module is (convoluted def coming up)

- $V/\mathbb{C}$ v.s. (probably ∞-dim')
- $\pi : \tilde{U}_\infty \to \operatorname{Aut}(V)$

s.t.

$$V = \bigoplus_{\substack{\rho \text{ fin-dim'l} \\ \text{irred cts} \\ \text{reps of } \tilde{U}_\infty}} V^\rho, \quad V^\rho = \sum_{\substack{\varphi : \rho \to V \\ \text{as } \tilde{U}_\infty\text{-morphism}}} \operatorname{im} \varphi$$

& $\forall\rho$ $\dim V^\rho < \infty$ $\forall \rho$.

($\tilde{U}_\infty$ is "max'l cpct subgp at ∞") Also (still def.)

- $(d\pi) : \underset{M_{2\times2}(\mathbb{R})}{\underset{\shortparallel}{\mathfrak{gl}_2(\mathbb{R})}} \xrightarrow{\mathbb{R}\text{-linear}} \operatorname{End}(V)$ s.t. 1) $(d\pi)[a,b] = [d\pi a, d\pi b]$, $d\pi$ $\mathbb{R}$ linear
  2) $(d\pi)\big|_{\text{Lie alg of } \tilde{U}_\infty} = \text{diff of } \pi$

Fig

This page left
blank due to
sheer incompetence.

& 3) $(d\pi)(uxu^{-1}) = \pi(u)\, d\pi(x)\, \pi(u)^{-1} \quad \forall x \in \mathfrak{gl}_2, u \in \tilde{U}_\infty$

You see, topology problems on $GL_2(\mathbb{R})$ mean we don't just define a $GL_2(\mathbb{R})$-module - we start with this big cpct subgp - its quite difficult to give an algebraic flavour that we like!

Define the Weil-gp $W_{\mathbb{R}} = < \mathbb{C}^\times, j \mid j^2 = -1, jzj^{-1} = \bar{z} \ \forall z \in \mathbb{C}^\times >$

$$0 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to C_2 \to 0$$

$\exists$ bijection $\left\{ \begin{array}{c} \text{irred admiss} \\ \text{rep}^{\text{n}}\text{s of } GL_2(\mathbb{R}) \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{cts (with usual tops) ss rep}^{\text{n}}\text{s} \\ \varphi : W_{\mathbb{R}} \to GL_2(\mathbb{C}) \end{array} \right\}$

$$\pi \longmapsto \varphi_\pi$$

Not saying much as they've got the same cardinality, clearly, probably. There is however a sensible way to do it.

Def$^{\text{n}}$ 1) $\varphi : W_{\mathbb{R}} \to GL_2(\mathbb{C})$ is <u>algebraic</u> if

$$\varphi\Big|_{\mathbb{C}^\times} : (z) \mapsto \begin{pmatrix} z^a \bar{z}^b & 0 \\ 0 & z^c \bar{z}^d \end{pmatrix} \quad , a,b,c,d \in \mathbb{Z}$$

2) An irred admissible $GL_2(\mathbb{R})$-module is <u>algebraic</u> if φ_π is alg.

An admissible $GL_2(\mathbb{A})$-module is

$V/\mathbb{C}$ v.s.

$\pi : GL_2(\mathbb{A}^\infty) \times \tilde{U}_\infty \to \mathrm{Aut}(V)$ s.t. if $U \subseteq GL_2(\mathbb{A}^\infty)$ is open then $V^U = \bigoplus_{\substack{\rho \text{ cts} \\ \text{fd irred} \\ \text{rep}^{\text{n}} \text{ of } \tilde{U}_\infty}} V^{U,\rho}$

$$\Big(- V^{U,\rho} = \sum_{\varphi : \rho \to V^U} \mathrm{Im}\, \varphi \Big)$$

& $\dim \forall V^{U,\rho} < \infty$

& $d\pi_\infty : \mathfrak{gl}_2(\mathbb{R}) \to \mathrm{End}(V)$ s.t. 1), 2), 3) as before

$\uparrow$

$\forall u \in GL_2(\mathbb{A}^\infty) \times \tilde{U}_\infty$

If π_v are irred admiss reps of $GL_2(\mathbb{Q}_v)$ for all places v of $\mathbb{Q}$ & π_v is unramified for almost all v, then $\hat{\check{\otimes}}\pi_v$ is an irred. admiss rep of $GL_2(\mathbb{A})$. If π is any irred. admiss rep of $GL_2(\mathbb{A})$, it factors $!$ly as $\hat{\check{\otimes}}\pi_v$ as above. π is algebraic $\overset{}{\underset{df}{\Leftrightarrow}}$ π_∞ is alg.

Def: Define a dif^l operator Δ on f^s on $GL_2(\mathbb{R})$ by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}$$

where we use coords z, x, y, θ by

$$z\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix}$$

(probably not ! rep^n here - but he guesses Δ doesn't mind)

Define $A = \{\varphi : {}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A}) \to \mathbb{C} \mid \text{erm}\}$

There's an action of $GL_2(\mathbb{A}^\infty) \times \tilde{U}_\infty$ via $g : \varphi(\_) \mapsto \varphi(\_g)$

Condition erm above is

1) $\varphi(\_u) = \varphi(\_)$ $\forall u \in U \subseteq GL_2(\mathbb{A}^\infty)$, open, cpct (depends on φ)

this is lots of functions

2) $\langle \varphi(\_\,\underset{u_\infty}{\cancel{\quad}}) \mid \forall u_\infty \in \tilde{U}_\infty \rangle_{\mathbb{C}}$ f.d.

2½) φ as a f^n on $GL_2(\mathbb{R})$ is smooth.

3) $\langle \{\Delta^i\varphi \mid i = 0,1,2\ldots\} \rangle$ is f.d.

4) φ is slowly increasing

A is an admissible $GL_2(\mathbb{A})$-module.
A isn't a direct sum of irreducibles (a bit upsetting)

Set $A^\circ = \{\varphi\in A \mid \varphi$ is cuspidal (ie $\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1&u\\0&1\end{pmatrix}g\right)du = 0 \ \forall g\in GL_2(\mathbb{A})\}$

A is the automorphic forms on GL_2

A° is the cuspidal automorphic forms

A° is a direct sum of irred:

Facts 1) $A^\circ = \bigoplus_{i\in I} \pi_i$, π_i irred admiss ($\pi_i \leftrightarrow$ grossen chars)

2) $\pi_i = \bigotimes_v \pi_{i,v}$, & $\pi_{i,v}\cong\pi_{j,v}$ for all but finitely many v, $\Rightarrow i=j$

(In prtic, $i\neq j \Rightarrow \pi_i\not\cong\pi_j$)

Recall we had too many grossen chars - χ was algebraic iff infinite part of χ was algebraic.

Defn The π_i are called cuspidal auto rep's of $GL_2(\mathbb{A})$
π_i alg $\Leftrightarrow$ $\pi_{i\infty}$ is alg.

Conjecture: every π_i alg $\rightsquigarrow$ compatible system of 2d-al rep's cm..

If π_i is an alg. cuspidal auto. rep. then

$$\pi_{i\infty} \text{ is } \begin{cases} D_k\otimes|\det|^{n+k} & k=1,2,3,\dots \\ & n\in\mathbb{Z} \\ M\otimes(\det)^n\otimes(\operatorname{sgn}\det)^\varepsilon, & \varepsilon=0,1,\ n\in\mathbb{Z}\end{cases}$$

$$D_k \leftrightarrow \quad z\mapsto\begin{pmatrix}z^{1-k}&0\\0&\bar z^{1-k}\end{pmatrix}|z|$$

$$j\mapsto\begin{pmatrix}0&1\\(-1)^{1-k}&0\end{pmatrix}$$

$M\mapsto \{$ triv rep of $W_{\mathbb{R}}$

If π is cuspidal automorphic $\left.\begin{matrix} \\ \end{matrix}\right\}$ then $\pi \otimes (\chi \circ \det)$ is cusp ~~alg~~ auto too.
χ is a grossenchar

Facts If $A^\circ = \bigoplus_{i \in I} \pi_i$, π_i irred, $\pi_i = \pi_i^\infty \otimes \pi_{i\infty}$

then $S_k = \bigoplus_{\substack{i \in I \\ \pi_{i\infty} = D_k}} \pi_i^\infty$, ie from S_k, the holo cusp form, we recover all alg cusp auto reps of $GL_2(\mathbb{A})$, except for when $\pi_\infty = M \otimes (\chi \circ \det)$

Interesting q: what on earth do ~~you~~ you do in M case?
In S_k case we can actually construct compatible systems of l adic rep's, or stg. In M case, we can't.

Principal conjectures:

Conj If π is an alg cusp auto rep of $GL_2(\mathbb{A})$, then there should exist a compatible system of 2-dim^l l adic reps

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(E_\lambda)$$

s.t if π_v is unramified & if $v \nmid N\lambda$ then ρ_λ is unramified at v, & $\rho_\lambda(\mathrm{Frob}_v)$ has char poly $(X - \alpha_v)(X - \beta_v)$, where π_v is a subquotient of $I(\chi_1, \chi_2)$ for a pair of unramified chars $\chi_1, \chi_2 : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ & $\alpha_v = \chi_1(\pi_v)$

$$\beta_v = \chi_2(\underset{\substack{\uparrow \\ \text{uniformiser of } \mathbb{Q}_v}}{\pi_v})$$

Moreover, each ρ_λ should be irred

One would also like to be able to go backwards

Rk: true if $\pi_\infty = D_k \otimes (\chi \circ \det)$

Now sthg on auto reps & how they're related to modular forms, or sthg.

Quaternion algebras - used by eg Ribet in the paper RT wants to talk about.

c) Quaternion Algebras

K a field. D/K is called a quaternion algebra if

Notes ≈200 reserved for this other stuff, which I understand now, & can't be bothered to fill in detail

1) D is a non-comm K-alg
2) K = centre of D
3) $[D:K]=4$
4) D simple (no non-triv 2-sided ideals)

e.g. $M_2(K)$ (200)

We call D split $\Leftrightarrow$ $D \cong M_2(K)$

If L/K is a field ext then $(D \otimes_K L)/L$ is a quat alg.

We say L splits D if $D \otimes_K L$ is split /L ie if $D \otimes_K L \cong M_2(L)$

If D is not split & $\alpha \in D \setminus K$, then $K(\alpha)$ = algebra generated by α

$K(\alpha)$ is commutative

$\therefore K(\alpha)$ is a field (can't be $K \oplus K$ as D is not split)

& $K(\alpha)$ splits D.

he's not sure why, but that's what the book says

If char $K=0$, & L/K is quadratic & splits D, then $L \cong K(\alpha)$ for some $\alpha \in D \setminus K$

(He guesses L/K separable quadratic & splits D is enough, no char 0)

Define $T: D \to K$
$N: D^\times \to K^\times$ thus:

If D split, $T = \mathrm{tr}$, $N = \det$

If not, & $\alpha \in D$, define

- $\alpha \in K \Rightarrow T\alpha = 2\alpha$, $N\alpha = \alpha^2$
- $\alpha \notin K \Rightarrow T\alpha = \mathrm{Tr}_{K(\alpha)/K}(\alpha)$, $N\alpha = N_{K(\alpha)/K}(\alpha)$

If L/K splits D,

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "T"'] & D \otimes L \cong M_{2\times 2}(L) \arrow[d, "\mathrm{tr}"] \\ K \arrow[r, hook] & L \end{tikzcd}$$

$$\begin{tikzcd} D^\times \arrow[r, hook] \arrow[d, "N"'] & (D\otimes L)^\times = GL_2(L) \arrow[d, "\det"] \\ K^\times \arrow[r, hook] & L^\times \end{tikzcd}$$

commute.

Probably a not-too-difficult exercise, (201)

T, N are the reduced trace & the reduced norm

$T(\alpha+\beta) = T(\alpha) + T(\beta)$ (202)
$N(\alpha\beta) = N(\alpha)N(\beta)$

(88)

eg 1) K alg closed
K finite } $\Rightarrow$ all quat algs are split.

2), $K = \mathbb{R}$: there are exactly 2 quat algs,

$$M_2(\mathbb{R}) \quad \& \quad \mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, z w \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

Tamper Razor wow ✓

3) K a local field (ie $K/\mathbb{Q}_p$ finite)

$\exists$ exactly 2 quat algs $/K$: $M_2(K)$ &

well → done Fran.

$$D_K = \left\{ \begin{pmatrix} z & w \\ \pi_K \, {}^{\sigma}w & {}^{\sigma}z \end{pmatrix} \middle| \; z w \in L \right\}$$

where L is the ! quad nr ext^n, $1 \neq \sigma \in \mathrm{Gal}(L/K)$.

L/K splits $D_K \Leftrightarrow 2 \mid [L:K]$

$\exists ! \; v_K : D_K^\times \to \frac{1}{2}\mathbb{Z}$ st $v_K(\alpha\beta) = v_K(\alpha) + v_K(\beta)$
$v_K(\alpha + \beta) \geq \min\{v_K(\alpha), v_K(\beta)\}$
v_K extends val^n on $K^\times$

in fact, $v_K = \frac{1}{2} v_K \circ N$

$\mathcal{O} \subset D$ is called an order of

- $\mathcal{O}$ is a free rank 4 $\mathcal{O}_K$-module (integers of K)
- $\mathcal{O}$ is an $\mathcal{O}_K$-algebra

K still local

An order is maximal if its ~~[illegible]~~ strictly not $\subseteq$ any strictly larger order.

If D is split ($\cong M_2(K)$) the max^l orders are exactly the conjugates of $M_2(O_K)$

If D is not split $\exists!$ max^l order $O_{D_K} = \{x \in D_K \mid v_K(x) \geq 0\}$

B

4) K a number field

If D/K is a quat. alg. set $S(D) = \{v \text{ place of } K \mid D_v =_{df} D \otimes_K K_v \text{ is not split}\}$

This provides an excellent invariant for D.

Facts
1) $S(D)$ is finite with an even # elts
2) $S(D)$!ly determines D
3) Any finite even set of places can arise. (apart from complex infinite ones)

eg $S(M_2(K)) = \emptyset$

D is definite if all infinite places are in $S(D)$
If not, D is indefinite.

12:30 tomorrow

Wed Recall • K no. field

$$\left\{\begin{matrix}\text{quat alg}\\ D/K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite even set}\\ \text{of places of } K\end{matrix}\right\}$$

$$D \longmapsto S(D) = \{v \mid D_v \text{ isn't split}\}$$

Orders: $O \subseteq D$ is called an order if
1) O is an O_K-algebra
2) O/O_K f.g. module
$O \otimes_{O_K} K \xrightarrow{\sim} D$

$\mathcal{O}$ is called a <u>max^l order</u> if it is not properly contained in any other order.

This is $\Leftrightarrow$ $\mathcal{O}_v$ is a max^l order in D_v $\forall v$ (exercise) ⟨203⟩
finite val's

Hint: $V/\mathbb{Q}$ say ; $L_0 \subseteq V$ a lattice
then other lattices $\leftrightarrow \{\Lambda_v \subset V \otimes \mathbb{Q}_v \mid \Lambda_v = (L_0)_v$ for all but finitely many $v\}$
finite v

Rk: max^l orders exist.

He's written "$M_2(K)^\times = GL_2(K)$" Now fix $\mathcal{O} \subseteq D$ max^l

Define $D^\times_{\mathbb{A}} = \{(x) \in \prod_{v \text{ (includes } \infty \text{ ones)}} D_v^\times \mid x_v \in \mathcal{O}_v^\times$ for all but finitely many $v\}$
[illegible] v

This does not in fact depend on the choice of the max^l order $\mathcal{O}$. ⟨204⟩

Put a topology on $D^\times_{\mathbb{A}}$ by saying $D^\times_\infty \times \prod_{v \text{ finite}} \mathcal{O}_v^\times$ is an open subgp with its usual topology. ⟨205⟩

Similarly $D^\times_{\mathbb{A}^\infty}$.

eg $D = M_2(K)$, $D^\times_{\mathbb{A}} = GL_2(\mathbb{A}_K)$, $D^\times_{\mathbb{A}^\infty} = GL_2(\mathbb{A}^\infty_K)$

$D^1 = \{x \in D^\times \mid Nx = 1\}$ eg $D = M_2(K) \rightsquigarrow D^1 = SL_2(K)$

<u>Facts</u> 1) If $v \notin S(D)$ then $D^1 D^1_v \subseteq D^1_{\mathbb{A}}$ is dense
obvious def^n: replace by ^{1}.

2) If D is not split
then $D^\times \backslash D^\times_{\mathbb{A}} / \prod_{v \mid \infty} K_v^\times$ is cpct
diagonally embedded

– erm this is probably right
He meant to say

$D^1 \backslash D^1_{\mathbb{A}}$ is cpct

Set $I \sim J$ if $I = \alpha J$ for some $\alpha \in D^\times$

Then $\left\{ \begin{matrix} \sim \text{ classes of} \\ \text{right ideals} \end{matrix} \right\} \leftrightarrow \left\{ D^\times \backslash D^\times_{\mathbb{A}^\infty} / \prod_v \mathcal{O}_v^\times \right\}$ (210)

↑ finite

Fix D, & a max^l order $\mathcal{O}_D$

Heinet: using $K=\mathbb{Q}$ here (or sth)

We can define admissible reps of $D_v^\times$ for all finite places v. (as before)

Exercise: if D_v is not split then all irred admiss reps are f.d. (211) (Hint? $D_v^\times / \mathbb{Q}_v^\times$ cpct)

If $D_v / \mathbb{Q}_v$ is not split then $\exists$ bijection

$$\left\{ \begin{matrix} \text{irred admiss} \\ \text{reps of } D_v^\times \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{irred admiss reps} \\ \text{of } GL_2(\mathbb{Q}_v) \text{ which} \\ \text{are not principal} \\ \text{series} \end{matrix} \right\}$$

(v finite)

So exclude these. (A principal series is $I(\chi_1, \chi_2)$, $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}_{\mathbb{Q}_v}$

or f.d. (ie 1-d) reps ie $\chi_0 \det$.

which is natural. eg $\chi|\cdot|_K^{1/2} \circ N \leftrightarrow \overset{\text{special}}{S}(\chi, \chi|\cdot|_K)$.

Fact: if π_v is an irred admiss rep^n of $D_v^\times$ for all finite v & $\pi_v^{\mathcal{O}_v^\times} \neq (0)$ for all but finitely many v, then we can define $\hat{\bigotimes}_v \pi_v$ which is an irred admiss. rep. of $D^\times_{\mathbb{A}^\infty}$. Any irred admiss rep of $D^\times_{\mathbb{A}^\infty}$ arises in this way, & determines the π_v !ly

↑ if you've done it for GL_2 then copy idea & get this. (212)

(NB better than the local finite measure results in classical case!) total?

For safety now, he'll fix $K=\mathbb{Q}$. Most of what he said go will probably go over, in some slightly more generalised form.

Fact 3) (Exercise) $\mathcal{D}^{\times} U GL_2^{+}(\mathbb{R}) = \mathcal{D}_{\mathbb{A}}^{\times}$

Same Strong(?) approximation

if $\infty \notin S(\mathcal{D})$

$U \subseteq \mathcal{D}_{\mathbb{A}^{\infty}}^{\times}$ open

$NU = \hat{\mathbb{Z}}^{\times}$

Hint: do the obvious thing.

Prove $N\mathcal{D}^{\times} = \mathbb{Q}^{\times}$. ⓶⓪⓺

erm he wants to tell us exactly when we can split $\mathcal{D}/K$ (K general no field for this bit - this should have been earlier)

A quadratic ext. L/K splits $\mathcal{D} \Leftrightarrow$ no place $v \in S(\mathcal{D})$ splits in L

Exercise Need this for 3).

4) (Exercise) If $\mathcal{D}$ is indefinite (ie $\infty \notin S(\mathcal{D})$) then all max^l orders in $\mathcal{D}$ are conjugate (Hint: $\mathcal{O}$ determined by $\mathcal{O}_v \; \forall v$) (207)

If $\mathcal{D}$ is definite (ie $\infty \in S(\mathcal{D})$) then $\exists$ bijection

'finite adeles' or sth

$$\left\{ \begin{matrix} \mathcal{D}^{\times} \text{ conj classes of} \\ \text{max}^l \text{ orders in } \mathcal{D} \end{matrix} \right\} \longleftrightarrow \left\{ \mathcal{D}^{\times} \backslash \mathcal{D}_{\mathbb{A}^{\infty}}^{\times} \Big/ \prod_{v \notin S(\mathcal{D})} \mathcal{O}_v^{\times} \times \prod_{v \in S(\mathcal{D})} \mathcal{D}_v^{\times} \right\}$$

(208)

Both are finite (geometry argument)

for $\mathcal{O}$ a fixed max^l order.

(NB may not be so easy if $K \neq \mathbb{Q}$. Put in $\mathbb{Q}_v^{\times}$ but we can take it out again)

5) (Exercise) A right ideal $I \subseteq \mathcal{O}$ ~~$I \subseteq$~~ some max^l order

Start again

A right ideal $I \subseteq \mathcal{D}$ for $\mathcal{O}$ (some max^l order)

is an $\mathcal{O}_K$-lattice s.t. 1) $I_v = \alpha_v \mathcal{O}_v$ for all finite places v

$\overset{\shortparallel}{\mathbb{Q}}$: $\mathcal{O}_K = \mathbb{Z}$

& 2) $I\mathcal{O} \subset I$ (NB 1) $\Rightarrow$ 2)) (209) (so I is a rt $\mathcal{O}$-module) (we want it locally principal)

$\leadsto I = \alpha \hat{\mathcal{O}} \cap \mathcal{D}$

$I\mathcal{O} = \alpha \hat{\mathcal{O}} \mathcal{O}_{\mathcal{D}} = \alpha \hat{\mathcal{O}} \cap \mathcal{D}$?

(27)

$S_k(\Gamma_j)$ f.d. $\therefore$ S_k^D is an admissible $D^\times_{\mathbb{A}^\infty}$ - module

Fact: S_k^D is a direct sum of irred $D^\times_{\mathbb{A}^\infty}$-modules

Thm (an "extraordinary fact") (- you're not getting anything new from this)

Let $S_k = \bigoplus_{i \in I} \pi_i$, π_i irred admiss.

Then $S_k^D = \bigoplus_{i \in I} \pi_i^D$ where

JL corresp

$$\pi_i^D = \begin{cases} (0) & \text{if for some } v \in S(D),\ \pi_{i,v} \text{ is principal series} \\ \widehat{\bigotimes_{v \notin S(D)}} \pi_{i,v} \otimes \bigotimes_{v \in S(D)} \pi_{i,v}^D \end{cases}$$

$\downarrow$ ($\Rightarrow GL_2(\mathbb{Q}_v) \cong D_v^\times$ - unique up to conjugation)

$\uparrow$ the rep corresponding to the non-principal-series rep $\pi_{i,v}$ (by previous fact)

Highly non-trivial fact

It can be translated down into elliptic modular forms:

we get some correspondence between forms on $\Gamma_1(N)$ & forms on Γ_j ← funny gps
& (Γ_j is v. weird - totally different from $\Gamma_1(N)$)
so we get a Hecke action on both & they're compatible etc. Weird.

He doesn't think that the proof would be spotted by anyone who hasn't seen the pf before

See the p48 top !!

Exercise: Fix N s.t. $p \nmid N$ $\forall p \in S(D)$. Let $U_1(N)^D = \prod_{v \in S(D)} \mathcal{O}_v^\times \times$ { hang on —

its $\left\{ x \in \prod_{v \notin S(D)} GL_2(\mathbb{Z}_v) \,\middle|\, x \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} (N) \right\}$

Let $M_S = \prod_{p \in S} p$, $\alpha_S \in GL_2(\mathbb{A}^\infty)$ s.t. $(\alpha_S)_v = \begin{cases} 1 & v \notin S \\ \pi_v & v \in S \end{cases}$

He wants to define spaces of modular forms on these objects!
Say $D \neq M_2(\mathbb{Q})$.

Assume $\infty \notin S(D)$ ie D indefinite. Fix $D_\infty^\times \xrightarrow{\sim} GL_2(\mathbb{R})$.

Set $S_k^D = \{ \varphi : D^\times \backslash D_{\mathbb{A}}^\times \to \mathbb{C} \mid$

1) $\varphi(\_u) = \varphi(\_)$ $\forall u \in U$ ie U open compact depending only on φ ($U \subseteq D^\times_{\mathbb{A}^\infty}$ I guess)
2) $\varphi(\_u_\infty) = \varphi(\_) j(u_\infty i)^{-k} (\det u_\infty)$ ($U_\infty = \mathbb{R}^\times SO_2(\mathbb{R})$ I guess)
3) $\forall g \in D^\times_{\mathbb{A}^\infty}$

$$\mathfrak{h}^\pm \to \mathbb{C}$$
$$hi \mapsto \varphi(gh) j(h,i)^k (\det h)^{-1} \quad \forall h \in GL_2(\mathbb{R})$$
(well-defined by 2) is holomorphic $\}$ (213)

He hasn't put in the slowly increasing criterion as this comes from cptness.

Exercise: $\varphi \in S_k^D \Rightarrow \varphi$ slowly increasing. (214)

He hasn't put in cuspidal cond as $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ has no analogue in D. This doesn't matter - they're "cuspidal" already. $\nexists$ analogue of modular forms in this set-up. (I guess we just need to check $\Gamma_j \backslash \mathfrak{h}$ is cpct - see below)

$U \subseteq D^\times_{\mathbb{A}^\infty}$ open cpct

$$D_{\mathbb{A}}^\times = \coprod_{i=1}^{r} D^\times g_i U GL_2^+(\mathbb{R}) \quad \text{for some } g_i. \ (215)$$

We have $(S_k^D)^U \xrightarrow{\sim} \bigoplus_{j=1}^{r} S_k(\Gamma_j)$

where $\Gamma_j = (D^\times \cap g_j U g_j^{-1}) \cap GL_2^+(\mathbb{R})$

2
1
6

The map is $\varphi \mapsto (f_j)$ where $f_j(hi) = \varphi(g_j h) j(h,i)^k (\det h)^{-1}$
for $h \in GL_2^+(\mathbb{R})$.

$\Gamma_j \subseteq GL_2^+(\mathbb{R})$
$\Gamma_j \underset{\text{commensurable}}{\sim} \mathcal{O}_D^\times \subset (D \otimes \mathbb{R})^\times = GL_2(\mathbb{R})$
commensurable - check things are discrete subgps, & that $\Gamma_j \backslash \mathfrak{h}$ is cpct (ie no cusps!)

It's a k-1-dim^l irred rep of $GL_2(\mathbb{C})$

(I think it's the ! one) (220) (up to twist)

could've put $\mathbb{C}$ but then def^n gets messy. It's all the same really

Set $S_k^D = \Big\{ \varphi : D^\times \backslash D_{\mathbb{A}}^\times \longrightarrow L_k(\mathbb{C}) \Big|$

D definite this time

$1)\ \varphi(\_u) = \varphi(\_)\ \forall u \in U,\ U \subseteq D^\times_{\mathbb{A}^\infty}$ open cpt depending only on φ

What will he do at ∞? $D_\infty/\mathbb{R}^\times$ is cpt, so cpt bit + centre at ∞ is whole thing

$2)$

Say $D_\infty = (U^D_\infty)^\infty$ Before, ~~U^∞~~ U^∞ or D_∞ a stly was abelian & life was easy so 2) was $\varphi(\_u_\infty) = \chi(u_\infty)\varphi(\_)$

If He'd put ~~[illegible]~~ cod$\varphi = \mathbb{C}$ then he'd put 2) $\langle \varphi(\_u_\infty) \mid u_\infty \in D_\infty^\times \rangle$ is $\cong \widehat{L_k(\mathbb{C})}$.

As cod$\varphi = L_k(\mathbb{C})$ he'll put

$$2)\ \varphi(\_u_\infty) = u_\infty^{-1}\varphi(\_)\quad \forall u_\infty \in D_\infty^\times \Big\}$$

$(K_\infty^\times)^\circ$

Exercise - check details if cod$\varphi = \mathbb{C}$.

Exercise (they're naturally iso) $= \Big\{ \varphi : D^\times_{\mathbb{A}^\infty} \longrightarrow L_k(\mathbb{C}) \;\Big|\; 1)\ \varphi(\_u) = \varphi(\_)\ \forall u \in U$ open cpt

$2)\ \varphi(\delta\_) = \delta\varphi(\_)\ \forall \delta \in D^\times \Big\}$ (221)

Fact S_k^D is an admissible $D^\times_{\mathbb{A}^\infty}$-module in the natural way:

action: $g(\varphi) = \varphi(\_g)$)

Simple eg $k=2$. $(S_k^D)^U = \Big\{ \varphi : \underbrace{D^\times \backslash D^\times_{\mathbb{A}^\infty}/U}_{\text{finite set}} \to \mathbb{C} \Big\}$

If $T \subseteq S(D)$ then $\exists$ map

$$\eta_T \quad S_k^{U_1(N) \cap U_0(M_{S(D)-T})} \longrightarrow S_k^{U_1(N) \cap U_0(M_{S(D)})}$$

(Exercise)

$$\varphi \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \alpha_T \end{pmatrix}(\varphi)$$

$S_k^{U_1(N) \cap U_0(M_{S(D)}), \text{new}}$ = orthog cmplmnt on $S_k^{U_1(N) \cap U_0(M_{S(D)})}$ of $\sum_{\substack{T \subseteq S(D) \\ T \neq \phi}} \text{Im}\, \eta_T$

The exercise is to show

$S_k^{U_1(N) \cap U_0(M_{S(D)}), \text{new}} \cong \left(S_k^D\right)^{U_1(N)^D}$ as modules for the Hecke operators T_p, S_p $\forall p \nmid M_{S(D)}$.

(& S_p not defined if $p \mid N$ so leave that out too)

- it's just a concrete version of the thm (218)

This was for an 'indefinite' quaternion algebra. $\boxed{k \geq 1}$

Now assume D definite, ie $\infty \in S(D)$. $\boxed{k \geq 2}$

Fix D_∞ - not split. $D_\infty \otimes_{\mathbb{R}} \mathbb{C}$ must be split as $\mathbb{C}$ alg closed

$\therefore$ $D_\infty \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} M_2(\mathbb{C})$ Fix an iso.

Let $L_k(\mathbb{C}) = S^{k-2}(\mathbb{C}^2)$ ie hgs polys of degree $k-2$ in 2 vars $/\mathbb{C}$.

$\uparrow$ Sym power

- a $k-1$ -dim^l v.s

As S^{k-2} is functorial, $L_k(\mathbb{C})$ inherits a natural action of $GL_2(\mathbb{C})$. (219)

For particular choices of U, this finite set is an analogue of a class gp.

Set $(S_k^D)^{triv} = \begin{cases} (0) & \text{if } k>2 \\ \{\varphi \in S_k^D \mid \varphi \text{ factors thru the map } \hat{D}^\times \backslash D_{\mathbb{A}}^\times \xrightarrow{N} \mathbb{Q}^\times \backslash \mathbb{A}^\times\} & (\text{if } k=2) \end{cases}$

Set $\overline{S_k^D} = S_k^D / S_k^{D,triv}$.

Still an admissible $D_{\mathbb{A}^\infty}^\times$-module ⓶⓶⓶

$\overline{S_k^D}$ (& S_k^D) are direct sums of irreds

<u>**Thm**</u> $(k\geq 2)$ Let $S_k = \bigoplus_{i\in I} \pi_i$, $\pi_i = \hat{\bigotimes} \pi_{i,v}$ irred

Then $\overline{S_k^D} = \bigoplus_{i\in I} \pi_i^D$

where $\pi_i^D = \begin{cases} (0) & \text{if for some } v\in S(D)\ \pi_v \text{ is princ. series} \\ \hat{\bigotimes}_{v\notin S(D)} \pi_{i,v} \otimes \bigotimes_{v\in S(D)} \pi_{i,v}^{D_v} & \text{otherwise} \end{cases}$

(explanation of symbols as before!)

Can't expect to see $k=1$ in this setting

NB exact sequence of admiss $D_{\mathbb{A}^\infty}^\times$-modules 1) splits !ly as $D_{\mathbb{A}^\infty}^\times$-mods

NB $S_k^D = \overline{S_k^D} \oplus S_k^{D,triv}$ 2)

(Challenging exercise)

- both sums of irreducibles

$k=2$: $S_k^{D,triv} = \bigoplus \chi \circ N$

$\chi : \mathbb{Q}^\times \backslash \mathbb{A}^{\infty\times} \to \mathbb{C}^\times$

- check)

Exercise: $S_k^{U_1(N)\cap U_0(M_{S(D)}),new} \cong (S_k^D)^{U_1(N)^D}$

as module for T_p & S_p for $p \nmid M_{S(D)}$

Exercise ~~(k≥2)~~ $S_2(\Gamma_0(p)) \cong \{\varphi : \{\sim\text{-classes of rt ideals of } \mathcal{O}_D\} \to \mathbb{C}\}$

when $S(D) = \{\infty, p\}$ & $\mathcal{O}_D$ is a max' order

× no! $D^\times$ conj classes of max' orders in D

- Also $\overline{S_k^D}$ contains no 1-d own rep's

(SAT)

∃ inner product on S_k^D i.e. sum of irreds.

Mon 17/2/92 Here's a correction to some exercise or other

$$U_1(N)^D \overset{\text{(new)}}{=} \prod_{v\in S(D)} \mathcal{O}_{D_v}^\times \times \left\{ x\in \prod_{v\notin S(D)} GL_2(\mathbb{Z}_v) \;\middle|\; x \equiv \begin{pmatrix} * & * \\ 0 & 1\end{pmatrix} (N) \right\}$$

then $D_v^\times \to D_v^\times/\mathcal{O}$ acts erm sthg Must make $D^\times$ act in a certain way

Easiest way to sort this out is change $\mathcal{O}_{D_v}^\times$ in def to $D_v^\times$, & take only $k=2$ in the 2 exercises which follow the last 2 thms

$$\begin{tikzcd}
GL_1: & \text{alg grossenchar} \arrow[r, hook] & \ell\text{-adic chars} \\
GL_2: & \text{alg cusp auto reps of } GL_2(\mathbb{A}^\infty) \arrow[r, dashed, leftrightarrow] & \text{2-dim}^{l}\ \ell\text{-adic reps} \\
& \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \\
& \text{those reps which occur } S_k \text{ (ch)} \arrow[r, dashed, leftrightarrow, "\circledast"] & \text{2-dim}^{l}\ \ell\text{-adic reps of } Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ st. } \rho(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}
\end{tikzcd}$$

$\uparrow$ complex conj

$\circledast \longrightarrow$ is known. He wants to talk about $\longrightarrow$.
We need some alg. geom. He's going to have to assume some.
Shimura's original $\to$ is less sophisticated - less alg geom & more chasing-about-points. This obscures the pt - given the sophisticated language of alg geom, it's easier.

$U \subseteq GL_2(\mathbb{A}^\infty)$

$$GL_2(\mathbb{Q}) \backslash \left(GL_2(\mathbb{A}^\infty)/U \times \mathfrak{h}^\pm \right)$$

$\uparrow$ acts diagonally

$h_i \longleftrightarrow h$

We can think of $\mathfrak{h}^\pm$ as $GL_2(\mathbb{R})/U_\infty$ so this latter object is

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / UU_\infty$$

(NT)

(same object still)

Or, $GL_2^+(\mathbb{Q}) \backslash \left(GL_2(\mathbb{A}^\infty)/U \times \underset{\underset{GL_2(\mathbb{R})/\tilde{U}_\infty}{\shortparallel}}{\mathfrak{h}} \right)$ (quotient out by $g\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)$)

- you see, $\mathfrak{h}$ has a cx structure that $GL_2(\mathbb{R})^+/U_\infty$ lacks

This is also $\coprod_{i=1}^{r} \Gamma_i \backslash \mathfrak{h}$, where :

(exercise)

(∃ nat map from here to whatever)

$$GL_2(\mathbb{A}^\infty) = \coprod_{i=1}^{r} GL_2^+(\mathbb{Q}) g_i U$$

& $\Gamma_i =_{df} GL_2^+(\mathbb{Q}) \cap g_i U g_i^{-1}$

$\Gamma_i \sim SL_2(\mathbb{Z})$.

Set M_U ~~M_{∞}~~ $= GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / UU_\infty$

abysmal curly M.

$\mathcal{M}$ M_U = complex pts of some smooth quasiprojective alg curve $\mathcal{M}_U/\mathbb{Q}$

$\overline{\mathcal{M}}$ $M_U \subseteq \overline{M_U}$ = compact RS

$\updownarrow$

$\overline{\mathcal{M}_U}$ = smooth projective alg. curve / $\mathbb{Q}$.

$S_2^U \leftrightarrow \Omega^1_{\overline{M_U}/\mathbb{C}}$ (differentials of $\overline{M}_U$)

So $S_2^U \hookrightarrow H^1(\overline{M}_U, \mathbb{C})$ (So mod forms somehow give us cohomology classes)

$\updownarrow$

$H^1(\overline{\mathcal{M}_U} \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ & ∃ Hecke action. Also ∃ action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on this latter H^1.

ie given $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ we get $H^1 \hookrightarrow H^1$.

$$\begin{tikzcd}[column sep=small] \lceil \text{Note} \quad H^1_{Betti}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \arrow[r, "\sim"] \arrow[d, "\text{choose } \mathbb{Q}_\ell \hookrightarrow \mathbb{C}"] & H^1_{ét}(\mathbb{Q}_\ell) \\ H^1_{Betti}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} =_{df} H^1_{Betti}(\mathbb{C}) & \end{tikzcd}$$

· inherits $\mathbb{Q}$-structure this way

Say A is a comm. ring. He could have said scheme.

he's not sure what curve means

E/A is an elliptic curve if E/A is a smooth projective curve of relative dimension 1, with connected fibres.

If 'curve' means 'scheme of relative dimension 1' then
1) E/A is a smooth proj curve with connected fibres...

2) E is a group scheme
ie ∃ alg morphisms $E \times_A E \xrightarrow{m} E$, $O: \operatorname{Spec} A \to E$ (ie "chosen pt on E") (an A-rational pt) origin

& $i: E \to E$ s.t. usual gp axioms are satisfied.
assoc, inverse etc

(eg $m(x, ix) = O$ ie

$$\begin{tikzcd} E \arrow[r, "1 \times i"] \arrow[d] & E \times E \arrow[r, "m"] & E \\ \operatorname{spec} A \arrow[urr, "O"'] & & \end{tikzcd}$$

commutes etc.

Rk: E is abelian NB E generically smooth + gp structure ⇒ E smooth
E proper ⇒ F ab.

In ptic A a field: E: $Y^2 Z = Z^3 P(X/Z)$, P is a cubic without repeated roots.

$E(A) = \{ (x:y:z) \mid \text{eqn holds, } x,y,z \text{ not all zero} \}$ $(x:y:z)$
$(x,y,z) = (\lambda x, \lambda y, \lambda z) \ \forall \lambda \in A^*$ ↓

Look at pts when $z \neq 0 \leftrightarrow \{ (x,y) \mid y^2 = p(x) \}$ $(\frac{x}{z} : \frac{y}{z} : 1)$

Also $z = 0 \rightsquigarrow (0:1:0)$, one more pt - the origin

If $N \in \mathbb{Z}_{>0}$ we get a map $N \cdot E \to E$ mult by N

$$x \mapsto \underbrace{x + \dots + x}_{n}$$

- an algebraic morphism (of schemes)
It preserves gp structure
ie a morphism of group schemes.

If A an alg closed field, it's onto

ker N makes sense if you think about ~~it~~ the right way

ker N / A is a finite flat gp scheme of order N^2

ie ∃ B/A ~~finite~~ algebra s.t. B locally free rank N^2/A ~~spec B is a gp scheme~~

eg if A is a field & char A ∤ N, this means Ker N is a finite gp $\cong (\mathbb{Z}/N\mathbb{Z})^2$ with a natural action of $\mathrm{Gal}(A^s/A)$

Non-canonically

$$(\ker N)(A) = \left((\mathbb{Z}/N\mathbb{Z})^2\right)^{\mathrm{Gal}(A^s/A)}$$

If A alg closed, then we're in even better shape finite flat gp schemes /A are just ab gps

We write $E[N]$ for this kernel.

He's thinking of A as an algebra, but ∃ analogues for when A is a scheme.

B/A : eg $B = \bigoplus_N A$, $(0,..,0,1,0..,0) = e_i$

$$\mu : B \to B \otimes_A B$$

$$e_i \mapsto \sum_{a+b \equiv i (N)} e_a \otimes e_b$$

$$m : \operatorname{Spec} B \times \operatorname{Spec} B \to \operatorname{Spec} B$$

This makes ~~B~~ spec B a finite flat gp scheme

We denote spec B by $(\mathbb{Z}/N\mathbb{Z})_A$

if you like, do it over $\mathbb{Z}$ & then get E/A by pullback.

$(\mathbb{Z}/N\mathbb{Z})^2_A$ is a finite flat gp scheme of order N^2,

An <u>elliptic curve with level N structure</u>

is $(E, \alpha) / A$ (A an algebra or a scheme)

E an elliptic curve $/A$, & an iso ~~of~~ $E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2_A$.

(∃ lots of α ie $|GL_2(\mathbb{Z}/N\mathbb{Z})|$ choices for a given E)

(NB ∃ problems of pts not defined over A or sthg: erm...

eg A a field of char 0

Need all N-division pts rational $/A$

& choose one of the $\#|GL_2(\mathbb{Z}/N\mathbb{Z})|$ isos of $E[N]$ with $(\mathbb{Z}/N\mathbb{Z})^2$)

<u>Fact</u> ∃ universal such object.

ie suppose $N \geq 3$. Then there is a quasiprojective smooth irred curve $M_N / \mathbb{Q}$ and an elliptic curve $\mathcal{E}_N \longrightarrow M_N$

ie Noetherian Scheme of rel dim 1 or sthg

on this is a scheme now

→ is read "over"

with a level N structure $\alpha_N : \mathcal{E}_N[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2_{M_N}$

s.t. if A is any $\mathbb{Q}$-algebra ($\mathbb{Q}$-scheme) & $(E, \alpha)/A$ is an elliptic curve with a level N structure then there is a ! map

$\operatorname{spec} A \to M_N$ s.t. (E, α) is the pullback of $(\mathcal{E}_N, \alpha_N)$ by $\operatorname{Spec} A \to M_N$

look it up in ~~Hartshorne~~ or sthg

In particular

$$\mathcal{M}_N(A) \overset{bij}{\longleftrightarrow} \left\{ \begin{array}{c} (E,\alpha)/A \text{ an elliptic curve} \\ \text{with level } N\text{-structure (up to iso)} \end{array} \right\}$$

Rk 1) $(\mathcal{E}_N, \alpha_N) / \mathcal{M}_N$ is unique.

2) If $M | N$ $\exists$ rational map $\mathcal{M}_N \to \mathcal{M}_M$

level M structure

$$\begin{tikzcd} \left(\mathcal{E}_N, \alpha_N\big|_{\mathcal{E}[M]}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathcal{E}_M, \alpha_M) \arrow[d] \\ \mathcal{M}_N \arrow[r, "\text{pullback: } \text{~~the map we~~ want.}"'] & \mathcal{M}_M \end{tikzcd}$$

The map $\mathcal{M}_N \to \mathcal{M}_M$ is étale.

$GL_2(\mathbb{Z}/N\mathbb{Z})$ acts on $\mathcal{M}_N$

$\mathcal{M}_M = \mathcal{M}_N / \ker\left(GL_2(\mathbb{Z}/N\mathbb{Z}) \to GL_2(\mathbb{Z}/M\mathbb{Z})\right)$, with action given below.

Take $\beta \in GL_2(\mathbb{Z}/N\mathbb{Z})$. He's gonna drop the $_N$'s now.

Over Rising, Washing us down

row vectors

$$\begin{tikzcd} \mathcal{E}[N] \arrow[r, "\alpha"] \arrow[dr] & (\mathbb{Z}/N\mathbb{Z})^2 \arrow[r, "\beta"] \arrow[d] & (\mathbb{Z}/N\mathbb{Z})^2 \arrow[dl] \\ & \mathcal{M}_N & \end{tikzcd}$$

(β acts via rt mult

$1 (\underbrace{\quad}_{N}) (\beta) : N$

I can't even be bothered.

$$\begin{tikzcd} (\mathcal{E}, \beta \circ \alpha) \arrow[r] \arrow[d] & (\mathcal{E}, \alpha) \arrow[d] \\ \mathcal{M}_N \arrow[r, "\beta^*"'] & \mathcal{M}_N \end{tikzcd}$$

the map we want.

Exercise $(\beta_1\beta_2)^* = \beta_2^* \beta_1^*$.

pullback

$$\begin{tikzcd} \text{NB } (\mathbb{Z}/N\mathbb{Z})^2 \arrow[d] & (\mathbb{Z}/N\mathbb{Z})^2_S \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\ \operatorname{spec}\mathbb{Z} & S \arrow[l, hook'] \end{tikzcd}$$

∴ define $\beta: (\mathbb{Z}/N\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/N\mathbb{Z})^2$
as morphism over spec $\mathbb{Z}$

$$\begin{tikzcd} \beta: (\mathbb{Z}/N\mathbb{Z})^2 \arrow[r] \arrow[dr] & (\mathbb{Z}/N\mathbb{Z})^2 \arrow[d] \\ & \operatorname{spec}\mathbb{Z} \end{tikzcd}$$

(Then $\exists ! \, \beta$ st on geometric fibres st. or even the generic fibre)

~~k alg closed, so~~ aargh.

s.t. on $\mathbb{Q}$-pts (ie morphism spec $\mathbb{Q} \to$ Homscheme)
β is just rt mult by the matrix β.

We want to end up with the adeles acting, really.

If $g \in GL_2(\mathbb{A}^\infty)$ & if $g^{-1} U_N g \subseteq U_M$, then define a map

$g^*: \mathcal{M}_N \to \mathcal{M}_M$ (thus?)

First suppose that $g \in M_{2\times 2}(\hat{\mathbb{Z}})$.

1) We get a map $(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\overset{\text{reduce mod } N}{g}} (\mathbb{Z}/N\mathbb{Z})^2$ by rt mult^n - a map of group schemes $/\mathbb{Z}$.
Let $A \subseteq \mathcal{E}[N]$ correspond to the kernel of g, via α.
Form $\mathcal{E}/A$, quotient, $/\mathcal{M}_N$, another elliptic curve.

2)
$$\begin{tikzcd} (\mathcal{E}/A)[M] \arrow[r, "\alpha", "\text{exercise - is iso.}"'] & M^{-1}A/A \arrow[d, "g"] \\ & (\mathbb{Z}/M\mathbb{Z})^2 \end{tikzcd}$$

ie $M^{-1}A$ is things in $(\mathbb{Z}/N\mathbb{Z})^2$ st when we multiply by M we're landin A

there's an exercise here: preimage of M div pt of N div pt because $g^{-1}U_N g \subseteq U_A$.
Also check $M^{-1}A = \{x \in (\mathbb{Z}/N\mathbb{Z}) \mid Mx \in A\}$
~~[illegible] $M^{-1}A$~~

We have ~~$(\mathcal{E}/A, g\circ\alpha)$~~ $(\mathcal{E}/A, g\circ\alpha)$ ← an elliptic curve level M structure

$$\begin{tikzcd} (\mathcal{E}/A, g\circ\alpha) \arrow[d] \\ \mathcal{M}_N \end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{E}/A, g\circ\alpha) \arrow[r] \arrow[d] & (\mathcal{E}, \alpha) \arrow[d] \\
\mathcal{M}_N \arrow[r, "g^*"'] & \mathcal{M}_M
\end{tikzcd}$$

g^* is the map we want

Ex 1) When both are defined, $(gh)^* = h^* g^*$

2) If $g \in GL_2(\hat{\mathbb{Z}})$ then you get the old def.

ie $g \in U_N \Rightarrow g$ acts trivially

$GL_2(\hat{\mathbb{Z}})/U_N \xrightarrow{\sim} GL_2(\mathbb{Z}/N\mathbb{Z})$ preserves $*$

So we're defined an action of $M_{2\times 2}(\hat{\mathbb{Z}})$

Ex If $\gamma \in \mathbb{Q}^\times_{>0} \cap M_2(\hat{\mathbb{Z}})$ then γ acts trivially. Hence one can extend the def to all of $GL_2(\mathbb{A}^\infty)$ by decreeing that $\mathbb{Q}^\times_{>0}$ acts trivially.

(Then $g \in GL_2(\mathbb{A}^\infty) \Rightarrow \exists \gamma \in \mathbb{Q}^\times_{>0}$ s.t. $\gamma g \in M_2(\hat{\mathbb{Z}})$.)

Tomorrow 1:30–3:00 Room in middle upstairs (257!)

18/2/92 There will be a lecture on Monday, come hell or high water (probably the latter)

Recall $(\mathcal{E}, \alpha)$, $\alpha : \mathcal{E}[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2_{\mathcal{M}_N}$

$$\begin{tikzcd}
(\mathcal{E}, \alpha) \arrow[d] \\
\mathcal{M}_N \arrow[d] \\
\operatorname{Spec} \mathbb{Q}
\end{tikzcd}$$

Recall of $A/\operatorname{Spec}\mathbb{Q}$

$$\begin{tikzcd}
(E, \alpha) \arrow[r] \arrow[d] & (\mathcal{E}, \alpha) \arrow[d] \\
A \arrow[r, "!"'] & \mathcal{M}_N
\end{tikzcd}$$

Also if $g \in GL_2(\mathbb{A}^\infty)$, $g^*: \mathcal{M}_N \to \mathcal{M}_M$, $(gh)^* = h^* g^*$

(N sufficiently large compared with g & M

$\mathcal{M}_N(\mathbb{C})$?

Well, say E/k is an ell. curve over a field.

Set $TE = \varprojlim_N E[N](\bar{k})$, where if $N|M$, $M/N : E[M] \to E[N]$

If char $k=0$, $TE \cong \hat{\mathbb{Z}}^2$ non-canonically.

If $T_\ell E = \varprojlim_r E[\ell^r](\bar{k})$, then $TE = \prod_\ell T_\ell E$.

Define $VE = TE \otimes_{\mathbb{Z}} \mathbb{A}^\infty$

Rks

1) $TE/NTE \cong E[N](\bar{k})$

2) If $\varphi: E \to E'$ is an isogeny (ie a non-trivial morphism of group schemes)
($\Rightarrow$ surjective on pts, finite kernel)

then $\varphi: TE \hookrightarrow TE'$

& $\varphi: VE \xrightarrow{\sim} VE'$

Consider pairs (E, A) (A isnt our algebra any more)
$E/\mathbb{C}$ an ell. curve
& $A: VE \xrightarrow{\sim} (\mathbb{A}^\infty)^2$

Say $(E,A) \sim (E',A')$ if $\exists \varphi: E \to E'$ an isogeny
& $u \in U_N = \{u \in GL_2(\hat{\mathbb{Z}}) \mid u \equiv 1 \ (N)\}$

st.

$$\begin{tikzcd}
VE \arrow[r, "A"] \arrow[d, "\varphi"] & (\mathbb{A}^{\infty})^2 \\
VE' \arrow[r, "A'"] & (\mathbb{A}^{\infty})^2 \arrow[u, "\text{mult by } u"']
\end{tikzcd}$$

Then $\exists$ bijection

$$\left\{ \begin{array}{l} \text{isoclasses of } (E,\alpha)/\mathbb{C} \\ \quad \text{ell curve with} \\ \quad \text{level } N \text{ structure} \end{array} \right\} \longleftrightarrow \left\{ (E,A)/_{\mathbb{C}} \right\} \Big/ \sim$$

(True in fact for any alg. closed field of char zero)

given by, if $(E,A) \in$ RHS, $\exists$ E_0 isogenous to E, s.t. $TE_0 \overset{A}{\longleftrightarrow} \hat{\mathbb{Z}}^2$. (ex)
Send (E,A) onto (E_0, α_0) where α_0 defined thus:

$$\begin{tikzcd}
E_0[N](\bar{k}) \arrow[r, "A"] \arrow[rr, bend right, "\alpha_0"'] & \frac{1}{N}\hat{\mathbb{Z}}^2 / \hat{\mathbb{Z}}^2 \arrow[r, "\sim"] & (\mathbb{Z}/N\mathbb{Z})^2
\end{tikzcd}$$

& conversely, given (E,α), send this to (E,A) where
$A \; TE \xrightarrow{\sim} \hat{\mathbb{Z}}^2$, determined up to $GL_2(\hat{\mathbb{Z}}^2)$ so far
& s.t. $\alpha \; A : E[N](\mathbb{C}) \cong TE/_{NTE} \xrightarrow{A} (\mathbb{Z}/N\mathbb{Z})^2$

(this determines A up to U_N)

<u>Ex</u> Check it gives a bij.
Nothing deep here - just a bookkeeping device.

Claim: $\exists$
hang on.

Claim $\exists$ bij $\;{}_{GL_2(\mathbb{Q})}\backslash^{GL_2(\mathbb{A})}/_{U_N U_\infty} \longleftrightarrow \mathcal{M}_N(\mathbb{C})$

Pf outline: LHS $\leftrightarrow {}_{GL_2(\mathbb{Q})}\backslash\left(GL_2(\mathbb{A}^\infty)/_{U_N} \times \mathfrak{h}^\pm\right)$ as before. This bijects with:

(✱)
$$\mathbb{C}^\times \Big\backslash \left\{ \Lambda \subseteq \mathbb{C},\ A \cdot \Lambda \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} (\mathbb{A}^\infty)^2 \;\middle|\; \Lambda \text{ 2-dim}^l\ \mathbb{Q}\text{-v.s.},\ \Lambda \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{C} \right\} \Big/ U_N$$

$z: \Lambda \mapsto \Lambda z$
$A \mapsto A.z^{-1}$

$A \mapsto u.A$

Trick of looking at $\mathbb{Q}$ spaces & not $\mathbb{Z}$-spaces is due to Deligne.

How: if $(g,\tau) \in GL_2(\mathbb{A}^\infty) \times \mathfrak{h}^\pm$, send this to

$$\langle \tau, 1 \rangle_{\mathbb{Q}},\ A: \langle \tau, 1 \rangle_{\mathbb{Q}} \otimes \mathbb{A}^\infty \xrightarrow{\sim} (\mathbb{A}^\infty)^2 \xrightarrow[\sim]{g} (\mathbb{A}^\infty)^2$$
$$\tau \longmapsto (1,0)$$
$$1 \longmapsto (0,1)$$

What needs to be checked: $g \rightsquigarrow \epsilon g U_N$ alters A by U_N ✓

$\gamma \in GL_2(\mathbb{Q})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $(\gamma g, \gamma\tau)$
$$\downarrow$$
$$\left(\langle \gamma\tau, 1 \rangle : \begin{array}{l} \gamma\tau \mapsto (1\,0)g \\ 1 \mapsto (0\,1)g \end{array} \right)$$
$$\wr$$
$$\left(\langle a\tau+b, c\tau+d \rangle,\ A: \lambda(a\tau+b)+\mu(c\tau+d) \mapsto (\lambda\ \mu)\gamma g \right.$$
$$\shortparallel$$
$$\langle \tau, 1 \rangle,\ A: (\lambda a+\mu c)\tau + (\lambda b+\mu d) \mapsto (\lambda a+\mu c, \lambda b+\mu d)g$$
$$\lambda'\tau + \mu' \mapsto (\lambda'\ \mu')g$$

- map well-defined

Certainly surjective

Exercise: injective

Now object (✱) bijects with $\left\{ E/\mathbb{C} \text{ ell. curve},\ A: VE \xrightarrow{\sim} (\mathbb{A}^\infty)^2 \right\}$ (which bijects with $\mathcal{M}_N(\mathbb{C})$ (as we saw earlier))
via given Λ, A, pick $\Lambda_0 \subseteq \Lambda$ a $\mathbb{Z}$-lattice, & send it to
$E = \mathbb{C}/\Lambda_0$, $VE = \Lambda_0 \otimes_{\mathbb{Z}} (\mathbb{A}^\infty) = \Lambda \otimes_{\mathbb{Q}} (\mathbb{A}^\infty) \xrightarrow{A} (\mathbb{A}^\infty)^2$. Exercise: bij. □

$$\begin{tikzcd}
\mathcal{M}_N(\mathbb{C}) \arrow[r, hook] \arrow[d, "g^*"] & {}_{GL_2(\mathbb{Q})}\backslash^{GL_2(\mathbb{A})}/_{U_N U_\infty} \arrow[d, "\text{rt mult by } g"] & h \arrow[d, mapsto] \\
\mathcal{M}_M(\mathbb{C}) \arrow[r, hook] & {}_{GL_2(\mathbb{Q})}\backslash^{GL_2(\mathbb{A})}/_{U_M U_\infty} & hg
\end{tikzcd}$$

exercise: check

Now if $GL_2(\hat{\mathbb{Z}}) \supseteq U \supseteq U_N$, set $\mathcal{M}_U = \overbrace{\mathcal{M}_N / \underset{\substack{\uparrow \\ \text{finite}}}{(U/U_N)}}^{\text{quotient scheme}} \quad (/\mathbb{Q})$

For U suff small the quotient makes sense eg $U \subseteq U_1(N)$, $N \geq 3$.

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\varphi"] \arrow[dr] & \mathcal{E}_2 \arrow[d] \\
& \mathcal{M}_{U_0(p) \cap U_N}
\end{tikzcd}$$

Ex

1) $\mathcal{M}_{U_0(p) \cap U_N}$ $(N, p) = 1$, $\mathcal{E}_1, \mathcal{E}_2$ ell curves w level N structure, φ an isogeny compatible with level N structure ker φ has order p

2) This has the obvious universal property

Next he wants to define sheaves on this object.

Def: We define a locally free sheaf $\mathcal{L}_n(\mathbb{Q}) / M_N$ (here $M_N =_{df} {}_{GL_2(\mathbb{Q})}\backslash^{(GL_2(\mathbb{A}^\infty)/_{U_N} \times \mathfrak{h}^\pm)}$,

$$\begin{array}{c}
{}_{GL_2(\mathbb{Q})}\backslash^{\left(GL_2(\mathbb{A}^\infty)/_{U_N} \times \mathfrak{h}^\pm \times S^n(\mathbb{Q}^2)\right)} \\
\downarrow \\
{}_{GL_2(\mathbb{Q})}\backslash^{\left(GL_2(\mathbb{A}^\infty)/_{U_N} \times \mathfrak{h}^\pm\right)} = M_N
\end{array}$$

He now wants to define an l-adic sheaf on $\mathcal{M}_N$, the arithmetic analogue to this one on M_N

$\mathcal{L}_n(\mathbb{Z}/\ell^r\mathbb{Z})$ étale sheaves on $\mathcal{M}_N$

$$\begin{tikzcd} \mathcal{M}_{N\ell^r} \arrow[d, "\text{étale, with Galois gp } \ker(GL_2(\mathbb{Z}/\ell^r\mathbb{Z}) \to GL_2(\mathbb{Z}/\ell^r\mathbb{Z}))"] \\ \mathcal{M}_N \end{tikzcd}$$

~~[illegible] (N already)~~

$GL_2(\mathbb{Z}/\ell^r\mathbb{Z}) \longrightarrow \mathrm{Aut}(S^n(\mathbb{Z}/\ell^r\mathbb{Z}))$

$(\ell^a \| N)$

One can define the l-adic sheaf as an inverse limit

Set $\mathcal{L}_n(\mathbb{Z}_\ell) = \varprojlim_r \mathcal{L}_n(\mathbb{Z}/\ell^r\mathbb{Z})$

(or go via $GL_2(\mathbb{Z}_\ell)$ & then take symmetric power) (comes down to same thing)

Rk: if $n=0$ we get cst sheaf (either setting!)

$H^1_{(c)}(\mathcal{M}_{U_N}, \mathcal{L}_n(\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. ($\mathbb{Q}_\ell$-vs)

(compact support) (if you like) (ie ∃ choice)

$\cong$

& $H^1_{(c),ét}(\mathcal{M}_N \times \operatorname{spec} \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

So compute it in the analytic setting, & get info about arithmetic setting.

The iso is because $\mathcal{M}_{N\ell^r}/\mathcal{M}_N$ defined /$\mathbb{Q}$ we get a cts action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (on?) $H^1_{(c),ét}(\mathcal{M}_N \times \operatorname{spec} \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$

$g \in GL_2(\mathbb{A}^\infty)$ $\quad g^{-1}U_N g \subseteq U_M$

rt mult on $GL_2(\mathbb{A}^\infty)/U_N$

$$\begin{tikzcd}
\mathcal{L}_n(\mathbb{Q}) \arrow[r, "g^*"] \arrow[d] & \mathcal{L}_n(\mathbb{Q}) \arrow[d] \\
M_{U_N} \arrow[r, "g^*", "(\text{rt mult})"'] & M_{U_M}
\end{tikzcd}$$

This induces $g_* : H^1_{(c)}(M_{U_M}, \mathcal{L}_n(\mathbb{Q})) \to H^1_{(c)}(M_{U_N}, \mathcal{L}_n(\mathbb{Q}))$

$(gh)_* = g_* h_*$ when both defined

If $N | M$, $H^1_{(c)}(M_M, \mathcal{L}_n(\mathbb{Q})) \xleftarrow[\sim]{U_N/U_M} H^1_{(c)}(M_N, \mathcal{L}_n(\mathbb{Q}))$

Set $H^1_{(c)}(\mathcal{L}_n(\mathbb{Q})) = \varinjlim_{N} H^1_{(c)}(M_N, \mathcal{L}_n(\mathbb{Q}))$

This has an action of $GL_2(\mathbb{A}^\infty)$, (All these objects are f.d.)
& becomes an
admissible $GL_2(\mathbb{A}^\infty)$-module.

There's a natural map $H^1_c \to H^1$
Set H^1_p = image of H^1_c in H^1.
) in any of these theories.

Of course, throughout this lit,

H^1 = cohomology
H^1_c = cohomology with cpct supports
H^1_p = parabolic coh.
= coh. of the interior

If A is a $\mathbb{Q}$-algebra, $H^1_{\underset{\uparrow}{(c,p)}}(\mathcal{L}_n(A)) = H^1_{(c,p)}(\mathcal{L}_n(\mathbb{Q})) \otimes A$.

choice $\emptyset, c, p$

He'd like to do the same thing in the arithmetic setting
(ℓ-adic coh)

Let $G^{\ell} = \{ g \in GL_2(\mathbb{A}^{\infty}) \mid g_\ell \in M_{2\times 2}(\mathbb{Z}_\ell) \}$ - a semigp he thinks

Suppose $g \in G^{\ell}$ & $g^{-1}U_N g \subset U_M$. Then we have

$g^* : \mathcal{M}_N \to \mathcal{M}_M$. We have $\mathcal{L}_n(\mathbb{Z}/\ell^a\mathbb{Z}) / \mathcal{M}_M$; we can

form its pullback $g^{**}\mathcal{L}_n(\mathbb{Z}/\ell^a\mathbb{Z}) / \mathcal{M}_N$, which is ~~giv~~ given by:

if $\ell^a \| M$, $U_N / U_N \cap g U_{M\ell^a} g^{-1} \longrightarrow \operatorname{Aut}(S^n(\mathbb{Z}/\ell^a\mathbb{Z}))$

$u \longmapsto g^{-1}ug \in$ ~~S~~ $GL_2(\hat{\mathbb{Z}})$

$\|$
Galois gp of some étale cover $\mathcal{M}'/\mathcal{M}_N$.

(Choose N' st. $U_{N'} \subseteq U_N \cap g U_{M\ell^a} g^{-1}$, then $\mathcal{M}'$ is a subcover of $\mathcal{M}_{N'}/\mathcal{M}_N$.)

Let $g : \mathcal{L}_n(\mathbb{Z}/\ell^s\mathbb{Z}) \to g^{**}\mathcal{L}_n(\mathbb{Z}/\ell^s\mathbb{Z})$

(s is st. $U_{N\ell^s} \subseteq U_N \cap g U_{M\ell^s} g^{-1}$, & $\ell^s \| N$)

be given by $S^n(\mathbb{Z}/\ell^s\mathbb{Z}) \xrightarrow{g} S^n(\mathbb{Z}/\ell^s\mathbb{Z})$

This gives maps $g_* : H^1_{(c,p)\,ét}(\mathcal{M}_M \times \operatorname{Spec}\overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$

pronounced "base change"

$\downarrow$

$H^1_{(c,p)\,ét}(\mathcal{M}_N \times \operatorname{Spec}\overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$

st $(gh)_* = g_* h_*$.

This was all on G^ℓ. Extend to whole thing:

~~If $\gamma \in GL_2^+(\mathbb{Q}) \cap G^\ell$ then γ~~

If $\gamma \in \mathbb{Q}^\times_{>0} \cap G^\ell$ then γ acts by γ^n & so if we tensor with $\mathbb{Q}_\ell$ we can extend to an action of $GL_2(\mathbb{A}^\infty)$ (by $\gamma \in \mathbb{Q}^\times_{>0}$ acting by γ^n)

The action here of $GL_2(\mathbb{A}^\infty)$ corresponds to the action on $H^1_{(C,P)}(M_N, \mathcal{L}_n(\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by the iso given before.

If $N \mid M$ then $\exists$ natural map

as its from obvious covering $M_N \to M_M$

$$(1_*:)\ H^1_{(C,P)\,\text{ét}}(\mathcal{M}_N \times \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell)) \hookrightarrow H^1_{(C,P)\,\text{ét}}(\mathcal{M}_M \times \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))^{\ker(GL_2(\mathbb{Z}/M\mathbb{Z}) \to GL_2(\mathbb{Z}/N\mathbb{Z}))}$$

1) If we $\otimes \mathbb{Q}_\ell$ this is iso
2) If we use H^1_c or H^1_p it's iso.
3) If $n > 0$ this is iso
4) If $n = 0$ then for fixed N the kernel is finite of order bounded indpt of M

($\exists$ spectral sequence H^r(finite gp) acts on H^0 : no H^0 on c $\Rightarrow$ no coh for c $\Rightarrow$ no coh for p

Define $H^1_{(C,P)}(\mathcal{L}_n(\overset{\mathbb{Z}_\ell}{\cancel{\mathbb{Q}_\ell}})) = \varinjlim_N H^1_{(C,P)\,\text{ét}}(\mathcal{M}_N \times \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$

Rk: $H^1_{(C,P)}(\mathcal{L}_n(\mathbb{Z}_\ell))$ is an admissible $GL_2(\mathbb{A}^\infty)$-module in the sense that 1) stabs of vectors are open
2) fixed pts of opens are finite $\mathbb{Z}_\ell$-modules.

$H^1_{(C,P)}(\mathcal{L}_n(\mathbb{Q}_\ell))$ has 2 (canonically iso) def's

$H^1_{(C,P)}(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_N} \Longleftarrow H^1_{(C,P)}(\mathcal{M}_N \times \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$ with finite cokernel.
Iso in cases listed above.

Also $H^1_{(C,P)}(\mathcal{L}_n(\mathbb{Z}_\ell))$ has an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which commutes with the action of G^ℓ & is cts (in some sense!)
-ie when restricted to $H^1_{(C,P)}(\mathcal{L}_n(\mathbb{Z}_\ell))^U$ for any open U. It's 3:07.

① (N7)

Mon /2/92

Recall $\mathcal{M}_N / \operatorname{spec} \mathbb{Q}$

$$\mathcal{M}_N(\mathbb{C}) = {}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A}) / U_N U_\infty$$

$\mathcal{L}_n(\mathbb{Q}) / \mathcal{M}_N(\mathbb{C})$ $\Big)$ recall defs
$\mathcal{L}_n(\mathbb{Z}_\ell) / \mathcal{M}_N$

& also $H^1(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$

$$\wr\wr$$

$$H^1(\mathcal{M}_N, \mathcal{L}_n(\mathbb{Z}_\ell)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

Recall H^1_c, $H^1_p = \operatorname{Im}(H^1_c \to H^1)$

Then $\varinjlim_N H^1(\mathcal{M}_N, \mathcal{L}_n(\mathbb{Z}_\ell)) = H^1(\mathcal{L}_n(\mathbb{Z}_\ell))$

$H^1(\mathcal{L}_n(\mathbb{Q}))$ ← admiss. $GL_2(\mathbb{A}^\infty)$-module

Also $H^1(\mathcal{L}_n(\mathbb{Q}_\ell))$ adm. $GL_2(\mathbb{A}^\infty)$-module,

& $H^1(\mathcal{L}_n(\mathbb{Z}_\ell)$ is preserved by $g \in GL_2(\mathbb{A}^\infty)$ if $g_\ell \in M_{2\times 2}(\mathbb{Z}_\ell)$

$H^1(\mathcal{L}_n(\mathbb{Z}_\ell))$ has a cts action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which commutes with the action of $GL_2(\mathbb{A}^\infty)$.

Same for H^1_c, H^1_p

Facts $\exists$ exact sequences $0 \to S_{n+2} \to H^1(\mathcal{L}_n(\mathbb{C})) \to \overline{M}_{n+2} \to 0$

$\overline{M}_{n+2}$ is M_{n+2} with usual action of $GL_2(\mathbb{A}^\infty)$ but with action of $\mathbb{C}$ changed:

if $\bar{f} \in \overline{M}_{n+2}$, $\lambda \bar{f} = \overline{(\bar{\lambda} f)}$ ie $\lambda \in \mathbb{C}$ acts via $\bar{\lambda}$

Similarly with H^1_c:

$$\begin{tikzcd}
0 \arrow[r] & S_{n+2} \arrow[r] & H^1(\mathcal{L}_n(\mathbb{C})) \arrow[r] & \overline{M_{n+2}} \arrow[r] & 0 \\
0 \arrow[r] & M_{n+2} \arrow[r] \arrow[u, "\text{proj}^n \text{ (use Petersson } \langle,\rangle)"'] & H^1_c(\mathcal{L}_n(\mathbb{C})) \arrow[r] \arrow[u] & \overline{S_{n+2}} \arrow[r] \arrow[u, "\text{inclusion}"'] & 0
\end{tikzcd}$$

The diagram commutes, & the maps preserve the action of $GL_2(\mathbb{A}^\infty)$

We then get

$$0 \to S_{n+2} \xrightarrow{i} H^1_p(\mathcal{L}_n(\mathbb{C})) \xrightarrow{\pi} \overline{S_{n+2}} \to 0 \quad \circledast$$

This is split. If we let complex conjugation c act on $H^1_p(\mathcal{L}_n(\mathbb{C}))$ $(= H^1_p(\mathcal{L}_n(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C})$ as $1 \otimes_{\mathbb{R}} c$, then

$c(iS_{n+2}) \xrightarrow[\sim]{\pi} \overline{S_{n+2}}$. So $\circledast$ is split.

This implies that $H^1_p(\mathcal{L}_n(\mathbb{C}))$ is a direct sum of irred. $GL_2(\mathbb{A}^\infty)$-modules, with the multiplicities at most 2. In fact we'll see in a minute that they're all 2.

Recall lots of silly facts about rep's (familiar for f.d. ones, & also true of ∞ dim' admissible)

Lemma V/E v.s, F/E field ext, $X \subseteq \mathrm{Hom}_E(V,E)$

Let $X^\perp = \{v \in V \mid x(v) = 0 \ \forall x \in X\}$

$X^\perp_F = \{v \in V \otimes_E F \mid x(v) = 0 \ \forall x \in X\}$ Then $X^\perp_F \cong X^\perp \otimes_E F$

Pf Reduce to f.d. case $\Box$

<u>Def</u>: An admis $GL_2(\mathbb{A}^\infty)$-module V is called <u>semi-simple</u> if $W \leq V$ a $GL_2(\mathbb{A}^\infty)$-submod $\Rightarrow V \cong W \oplus (V/W)$

<u>Lemma</u> 1) V/E admis $GL_2(\mathbb{A}^\infty)$-module, F/E a field ext, $V \otimes_E F$ ss $\Rightarrow$ V is ss

2) V/E (irred) admis $GL_2(\mathbb{A}^\infty)$-mod, F/E field ext $\Rightarrow \mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V) \otimes F \cong \mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V \otimes F)$

<u>Pf</u> Easy exercise

<u>Lemma</u> If V is an admis ss $GL_2(\mathbb{A}^\infty)$-module then quotients & submodules are ss.

<u>Pf</u> Suppose $0 \to W \to V \to U \to 0$

1) $X \subset U$ submod, $\tilde{X}$ = preimage, $V = \tilde{X} \oplus U/X$ so $U = X \oplus U/X$

2) $X \subset W$ submod, $V = X \oplus (V/X)$, $V/X \supseteq W/X$ by 1) $V = X \oplus W/X \oplus V/W$

W corresponds to $X \oplus W/X$ □

<u>Lemma</u> V/E ss admis $GL_2(\mathbb{A}^\infty)$-mod. Then $V \cong \bigoplus$(irred admis $GL_2(\mathbb{A}^\infty)$-mods)

<u>Pf</u> Let $U_1 \supseteq U_2 \supseteq \ldots$ be open cpct subgps s.t. $\bigcap_i U_i = (1)$

Define inductively admis reps π_i, V_i s.t. 1) π_i irred

2) $V = \pi_1 \oplus \ldots \oplus \pi_i \oplus V_i$

3) $\forall j \, \exists i_0 \, \forall i \geq i_0 \; V_i^{U_j} = (0)$

i.e. sort out all fixed pts

Then $V \cong \bigoplus_i \pi_i$. So how do we do it?

Define π_{i+1}, V_{i+1} thus: Let j be minl s.t. $V_i^{U_j} \neq (0)$.

Choose $W \leq V_i$ a $GL_2(\mathbb{A}^\infty)$ submod, minl subject to $W^{U_j} \neq (0)$

Take $\pi_{i+1} = W$, $V_{i+1} = V_i/W$

Exercise: this works

2 more general lemmas

Lemma Suppose V/F is a ss admiss $GL_2(\mathbb{A}^\infty)$-module.

Then i) V is irred. $\Leftrightarrow$ $\mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V)$ is a division algebra, f.d. $/F$

ii) V abs irred $\Leftrightarrow$ $\mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V) = F$

ie $V \otimes_F E$ irred $\forall$ exts E/F

Pf All as for f.d case, given 1 observation.

i) $\Rightarrow$: Choose U open cpct s.t. $V^U \neq (0)$. Then (ex) $\overset{D}{\overset{\shortparallel}{\mathrm{End}}}_{GL_2(\mathbb{A}^\infty)}(V) \hookrightarrow \mathrm{End}_F(V^U)$ & so are f.d. $/F$ & so is Artinian (DCC)

Let J = Jacobson radical of D = $\bigcap$ all max^l 2-sided ideals

Then J is nilpt (general th^m)

$\therefore J^n = 0 \therefore J^n V = 0 \therefore JV \neq V \therefore JV = (0)$ as V irred $\therefore J = (0)$

Also, if $e \in D$, $e^2 = e$, then eV is a submod $\therefore eV = V$ or 0 $\therefore e = 1$ or 0

Artinian ring, no non-triv idempotents, $J = 0 \Rightarrow D$ is a division algebra.

($\Leftarrow$) ex

ii) Whichever way round, we know V irred. Let $D = \mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V)$

Then V abs irred $\Leftrightarrow$ $D \otimes_F E$ division alg $\forall E/F$ by (i)
$\Leftrightarrow$ $D = F$ by a well-known fact on division algebras.

One final general lemma. Oh - make that 2

Lemma Suppose V/F is an irred admiss $GL_2(\mathbb{A}^\infty)$-module. Let $Z =$ centre of $\mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V)$ & let $[\mathrm{End}_{GL_2(\mathbb{A}^\infty)}(V) : Z] = d^2$.

Then $V \otimes_F \bar{F} = \bigoplus_{\sigma \in I} W_\sigma^d$ (ie W_σ to the d^{th} power)

when $I = \{\sigma : Z \hookrightarrow \bar{F}\}$

$$\begin{tikzcd} Z \arrow[rr, hook] & & \bar{F} \\ & F \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

(note Z/F finite)

& $W_\sigma = V \otimes_{Z,\sigma} \bar{F}$ (embed Z via σ)

W_σ is abs irred (endomorphisms $= \bar{F}$)

Secondly if $\tau \in \mathrm{Gal}(\bar{F}/F)$ then $\tau W_\sigma \cong W_{\tau\sigma}$

Pf Exercise □

One final general lemma

Lemma If V_1, V_2 are irred admiss $GL_2(\mathbb{A}^\infty)$ modules$/F$, & if E/F is a field, then $V_1 \cong V_2 \Leftrightarrow V_1 \otimes_F E \cong V_2 \otimes_F E$

Pf Ex.

After all these generalities we can now return to the general sit at hand (as true for $\mathbb{C}$)

Cor 1) $H^1_p(\mathcal{L}_n(\mathbb{Q}))$ is a ss admiss rep of $GL_2(\mathbb{A}^\infty)$

not an evident fact

2) If π is an irred admiss rep occuring in S_k, $k \geq 2$, then $\exists$ number field E & an admiss rep π' over E s.t. $\pi = \pi' \otimes_E \mathbb{C}$.

3) In ptic, if π is as above, it's defined $/\overline{\mathbb{Q}}$ so we can set E_π = fixed field of $\{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma\pi \cong \pi\}$. Then E_π is a number field, & $\exists \pi^\circ$ an irred admiss rep of $GL_2(\mathbb{A}^\infty)$ over $\mathbb{Q}$ s.t. E_π is the centre of $End_{GL_2(\mathbb{A}^\infty)}(\pi^\circ)$. Moreover, the endomorphisms have degree 1 or 4 over E_π.
Also $\pi = \pi^\circ \otimes_{E_\pi} \mathbb{C}$ $\square$

He wants to now show that in fact it's always degree 1, & this'll need considerably more work: cohomology & twisting by chars etc.

Lemma If π is an irred admiss constituent of S_k then $^c\pi \cong \|\cdot\|^{2-k}\check{\pi} \cong \|\cdot\|^{2-k}\chi_\pi^{-1}\pi$

complex conj.

H.it because of inner product on S_k

central char

Pf Exercise. Use strong multiplicity 1

Cor E_π is totally real or a CM field $\Big\{$ ie $\forall \sigma : E_\pi \hookrightarrow \mathbb{C}$ $c \circ \sigma E_\pi = \sigma E_\pi$, & $\sigma^{-1} c \sigma : E_\pi \to E_\pi$ is indpt of σ

good def. of CM in this case! (or totally real)

equiv to E_π totally real or a purely imag quad ext of a totally real field

Now he's going to need a pairing on the cohomology.

$\exists$ pairing $S^n(R^2) \times S^n(R^2) \to R$, R any ring, defined thus:

take the standard basis e_1, e_2 of R^2
Take corresp. standard basis of $S^n(R^2)$ $\{e_1^{\otimes a} \otimes e_2^{\otimes n-a}\}$

& if $x, y \in S^n(R^2)$, set $\langle x, y\rangle =_{df} {}^t x\, S^n(w)\, y$ $\quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

row vector

col vector wrt basis

Exercise$^{1)}$ if $n!$ is invertible in R then this is a perfect pairing

Maybe$^{1k)}$: Exercise: This is a perfect pairing

2) $\alpha \in GL_2(R)$ then $\langle \alpha x, \alpha y \rangle = (\det' \alpha)^n \langle x, y \rangle$ 3) $\langle x,y \rangle = (-1)^n \langle y, x \rangle$

(functoriality of) perfect, by Poincaré duality,

Now 2) ⇒ induces a pairing on the cohomology

$$\begin{tikzcd} H^1_c(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q})) \times H^1(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q})) \arrow[r] & H^2_c(\mathcal{M}_N(\mathbb{C}), \mathbb{Q}) \arrow[d, "\text{so-called trace map}"'] \\ & \mathbb{Q} \end{tikzcd}$$

Also $H^1_p(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q})) \times H^1_p(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q})) \to H^2_c(\mathcal{M}_N(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}$ [illegible]

not ?! H^2_p vanishes!!

These induce

$$H^1_p(\mathcal{L}_n(\mathbb{Q})) \times H^1_p(\mathcal{L}_n(\mathbb{Q})) \to \mathbb{Q} \quad \text{st}$$

1) $\langle gx, gy \rangle = \|\det g\|^{-n} \langle x, y \rangle$ for $g \in GL_2(\mathbb{A}^\infty)$

2) $\langle\ ,\ \rangle$ restricted to a finite level $H^1_p(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q}))$ gives the previous pairing up to a scale:

$$\begin{tikzcd} \text{if } N|M, & H^1_c(N) \times H^1(N) \arrow[r] \arrow[d] & H^2_c(N) \arrow[d] \\ & H^1_c(M) \times H^1(M) \arrow[r] & H^2_c(M) \end{tikzcd} \quad \text{commutes}$$

$$\begin{tikzcd} H^2_c(N) \arrow[r, "tr"] & \mathbb{Q}. \\ H^2_c(M) \arrow[u, "tr"] \arrow[ur, "tr"'] & \end{tikzcd} \quad \text{commutes}$$

$$\begin{tikzcd} H^2_c(N) \arrow[d, bend right, "res"'] \\ H^2_c(M) \arrow[u, bend right, "tr"'] \end{tikzcd}$$

& $\circlearrowleft$ is mult by the degree of the cover $\mathcal{M}_M \downarrow \mathcal{M}_N$

3) $\langle x, y\rangle = (-1)^{n+1} \langle y, x\rangle$

4) $\otimes \mathbb{Q}_\ell$ Everything has an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
&

$$\langle \sigma x, \sigma y\rangle = \underset{\substack{\uparrow \\ \text{cyclotomic}}}{\chi_\ell}(\sigma)^{-(n+1)} \langle x, y\rangle$$

Rk $H^1_p(\mathcal{L}_n(\mathbb{Q})) = \oplus\, \pi_i$ π_i irred admiss / $\mathbb{Q}$

If ~~π~~ $\exists\, x \in \pi_i$, $y \in \pi_j$ with $\langle x, y\rangle \neq 0$, then $\pi_i \cong \pi_j$

(Pf. $\pi_i \longrightarrow \|\det\|^{-n} \check{\pi}_j$ non triv HM $\therefore$ iso

$\cong {}^{\iota}\pi_j \cong \pi_j$ as it's defined / $\mathbb{Q}$)

If π is irred constituent of S_k &

$$\chi \quad \mathbb{Q}^\times_{>0} \backslash \mathbb{A}^{\infty\times} \longrightarrow \mathbb{C}^\times$$

then $\pi \otimes \chi = \{ f \otimes \chi : g \mapsto f(g)\, \chi(\det g) \mid f \in \pi \}$

$\cancel{\cong}\, \pi \otimes \chi$ is also a constituent of S_k

Over $\mathcal{M}_N$ we get an iso $\mathbb{Z}/N\mathbb{Z} \Rightarrow \underset{e_1 \wedge e_2}{\mu_N}$ This comes from $e_1 \wedge e_2$

$$\begin{tikzcd} (\mathbb{Z}/N\mathbb{Z})^2 & \text{Take } \Lambda^2 & \Lambda^2(\mathbb{Z}/N\mathbb{Z})^2 \arrow[r, "\sim"] & \mathbb{Z}/N\mathbb{Z} \\ \mathcal{E}[N] \arrow[u, "\alpha", "\wr"'] & & \Lambda^2 \mathcal{E}[N] \arrow[u, "\alpha", "\wr"'] \arrow[r, "\sim"] & \mu_N \text{ (Weil pairing)} \end{tikzcd}$$

$\mathcal{M}_N \times \operatorname{spec} \overline{\mathbb{Q}} = \coprod_{\substack{\zeta \text{ a primitive} \\ N^{\text{th}} \text{ root of } 1}} M_{N,\zeta}$, $M_{N,\zeta}$ = those pts where $1 \mapsto \zeta$

Exercise: ~~$M_{N,\zeta}$~~ $\mathcal{M}_{N,\zeta}(\mathbb{C})$ is connected

$\mathcal{M}_{N,\zeta}/\bar{\mathbb{Q}}$ is irred

There's sthg essentially scheme-theoretic going on

abs irred
$$\begin{tikzcd}
\mathcal{M}_N \arrow[d] \\
\operatorname{Spec}\mathbb{Q}(\zeta_N) \arrow[d] \\
\operatorname{Spec}\mathbb{Q}
\end{tikzcd}$$

i.e. $\begin{tikzcd}\mathcal{M}_N \arrow[d] \\ \operatorname{Spec}\mathbb{Q}\end{tikzcd}$ factors thru' $\operatorname{Spec}\mathbb{Q}(\zeta_N)$

Hence $H^0(\mathcal{M}_N(\mathbb{C}), \mathbb{Q}) \cong \operatorname{Maps}\left(\left\{\begin{smallmatrix}\text{primitive}\\ N^{\text{th}}\text{ roots of}\\ \text{unity}\end{smallmatrix}\right\} \to \mathbb{Q}\right)$

The Galois action on $H^0(\mathcal{M}_N, \mathbb{Q}_\ell) \cong \operatorname{maps}\left(\left\{\begin{smallmatrix}\text{primitive}\\ N^{\text{th}}\text{ root}\\ \text{of } 1\end{smallmatrix}\right\} \to \mathbb{Q}_\ell\right)$

is $\sigma(f)(\zeta) = f(\sigma^{-1}\zeta)$

Final (for today) exercise:

Ex Limit! $H^0(\bar{\mathbb{Q}}) = \displaystyle\bigoplus_{\chi : \mathbb{Q}^\times_{>0}\backslash(\mathbb{A}^\infty)^\times \to \bar{\mathbb{Q}}^\times} \chi\circ\det$ as a $GL_2(\mathbb{A}^\infty)$-module

$$H^0(\bar{\mathbb{Q}}_\ell) \cong \bigoplus_{\substack{\chi \text{ as}\\ \text{above}}} (\chi\circ\det)\otimes(\underset{\substack{\uparrow\\ \ell\text{-adic Galois character assoc to } \chi}}{\chi_\ell^G})^{-1}$$

as a $GL_2(\mathbb{A}^\infty) \times \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module.

Richard has had considerable trouble understanding the pf of the Eichler - Shimura relations.

He's got it now, modulo 2 facts:

i) comparison of étale coh ~~mod~~ char 0 & p (he'll talk about this later)

ii) normalisations gone wrong (he hasn't had time to sort out the error)

Reference for this stuff is Deligne's article in Sem. Bourbaki 1969, Fraser says it's # 355. (LNMS vol 179?)

Springer Shimura

Recall $H^1_p(\mathcal{L}_n(\mathbb{Q}_p))^2 \longrightarrow H^2_c(\mathcal{L}_{n}(\mathbb{Q}_p)) \xrightarrow{tr} \mathbb{Q}_p$

Pairing behaves well wrt action of $GL_2(\mathbb{A}^\infty) \times Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Perfect when we take U-invts, $U \subseteq GL_2(\mathbb{A}^\infty)$ cpct open subgp.

$$H^0(\overline{\mathbb{Q}}_\ell) = \bigoplus_{\chi : \mathbb{Q}^\times_{>0} \backslash \mathbb{A}^{\infty\times} \to \overline{\mathbb{Q}}_\ell^\times} (\chi \circ \det) \otimes (\chi_\ell^G)^{-1}$$

cts wrt discr top. (He said 'Global CFT')

Now fix π occurring in S_k $(k \geq 2)$

Let π have central character χ_π

Set $\chi = \chi_\pi \|\cdot\|^{k-2}$ $\mathbb{Q}^\times_{>0} \backslash (\mathbb{A}^\infty)^\times \rightarrow E_\pi^\times$

Choose $x \neq 0$ in $H^0(\overline{\mathbb{Q}}_\ell)$ in the $\chi \circ \det$ eigenspace.

(WLOG $x \in H^0(E_\pi)$ - (to get the full decomp we'd really have to go to $\overline{\mathbb{Q}}$, or sthg))

We define

$$H^1_p(\mathcal{L}_n(E_\pi))(\pi)^2 \longrightarrow E_\pi$$

$\uparrow$

Σ of the images of all $GL_2(\mathbb{A}^\infty)$-morphisms π to $H^1_p(\mathcal{L}_n(E_\pi))$

μ_0 μ_1

by $(a,b) \longmapsto \langle a, x \cup b\rangle$

$\uparrow$ cup product

(If we'd taken the 'obvious' pairing it'd be degenerate)

Ex 1) The pairing is non-degenerate

2) $(\sigma a, \sigma b) \mapsto ((\chi_\pi)^G_\lambda\, \chi_\ell)^{-1}(\sigma)\, \langle a, x \cup b\rangle$

~~$\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$~~ $\quad \shortparallel$ $\chi^G_{\pi,\lambda}$ (notation)

Then $\mathrm{End}\, H^1_p(\mathcal{L}_n(E_\pi))(\pi) = D$, where D is some quaternion algebra, centre E_π.

For each prime λ of E_π we get a cts rep^n

$$\rho_{\pi,\lambda} : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to D^\times_\lambda$$

The pairing $\Rightarrow$ $N_0\, \rho_{\pi,\lambda} = (\chi^G_{\pi,\lambda}\ \chi_\ell)^{-1}$

$\uparrow$ a reduced norm N. (If split it's = det) $\quad$ cyclo

If $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a complex conjugation,

$\chi^{G}_{\pi,\lambda}(c) = 1$ (exercise unravel def^n s!)

$\Rightarrow N_0\,\rho_{\pi,\lambda}(c) = -1$

(He wrote $\rho_{\pi,\lambda}(c)^2 = 1 \Rightarrow c = \pm 1$ or D_λ is split, then rubbed it off!)

c acts ctsly on $M_N(\mathbb{C})$ & so acts (on!) $H^1_p(\mathcal{L}_n(E_\pi))$

This action is compatible with the action on $H^1_p(\mathcal{L}_n(E_{\pi,\lambda}))$

Thus $\exists\ \rho_\pi(c) \in D^\times$ st $\rho_\pi(c)$ is conj to $\rho_{\pi,\lambda}(c) \in D^\times_\lambda \quad \forall \lambda$

Thus

$$\left.\begin{array}{l} 1)\ N_0\,\rho_\pi(c) = -1 \\ 2)\ \rho_\pi(c)^2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \rho_\pi(c) = \pm 1 \ \#\!\!\!\!/ \\ \text{or } \rho_\pi(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$\Downarrow$

D split, as $(\rho_\pi(c)+1)(\rho_\pi(c)-1) = 0$
$\Rightarrow \rho_\pi(c) = \pm 1$ if D is a division algebra.

Sum everything up

Lemma Suppose π occurs in S_k, $k \geq 2$. We get π^0, an admissible irred rep of $GL_2(\mathbb{A}^\infty)$ over $\mathbb{Q}$ s.t. $\pi^0 \otimes \mathbb{C}$ contains π. Let $D_\pi = \mathrm{End}_{GL_2(\mathbb{A}^\infty)}(H^1_p(\mathcal{L}_{k-2}(\mathbb{Q}))(\pi^0))$. Then

$D_\pi \cong M_2(E_\pi)$, $E_\pi/\mathbb{Q}$ a no. field.

For each prime λ of E_π there's a cts rep^n

$\rho_{\pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(E_{\pi,\lambda})$

st $\det \rho_{\pi,\lambda} = (\chi^{G}_{\pi,\lambda}\,\chi_\ell)^{-1}$ $\square$

Next understand Trace at Frob_p.

We'll relate it to T_p as defined earlier.

We'll set up another way of looking at things

Fix N, p st. $p \nmid N$

Let $\mathcal{M}_{N,p}$ correspond to $U_N \cap U_0(p)$.

In ptic, $\mathcal{M}_{N,p} = \mathcal{M}_{Np}/\mathbb{F}_p$

Set $B = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_2 \}$

$$\begin{tikzcd}
\mathcal{E} \arrow[d] & \alpha \; \mathcal{E}[Np] \arrow[r, "\sim"] \arrow[d, phantom, "\wr\wr"] & (\mathbb{Z}/Np\mathbb{Z})^2 \arrow[d, phantom, "\wr\wr"] \\
\mathcal{M}_{Np} & \alpha_N \times \alpha_p \; \mathcal{E}[N] \times \mathcal{E}[p] \arrow[r, dashed] & (\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/p\mathbb{Z})^2
\end{tikzcd}$$

Let $C = \alpha_p^{-1} \{ (0, x) \mid x \in \mathbb{Z}/p\mathbb{Z} \}$

$C \subseteq \mathcal{E}[p]$ is a finite flat group scheme of order p

(it's actually $(\mathbb{Z}/p\mathbb{Z})$)

Over $\mathcal{M}_{N,p}$ we have $\alpha : \mathcal{E}[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$

$C \subseteq \mathcal{E}[p]$ f.f. gp scheme order p.

$\therefore$ we have

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{E}_2 \arrow[dl] \\
& \mathcal{M}_{N,p} &
\end{tikzcd}$$

φ an isogeny, $\ker\varphi$ order p

$\mathcal{E}_1 = \mathcal{E}, \quad \mathcal{E}_2 = \mathcal{E}/C$

$\alpha . \; \mathcal{E}_1[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$

In fact this is universal i.e.

$$\begin{tikzcd}
E_1 \arrow[r, "\varphi"] \arrow[dr] & E_2 \arrow[d] \\
& S
\end{tikzcd}
,\quad
\begin{tikzcd}
E_2[N] \arrow[r, "\sim", "\alpha"'] \arrow[dr] & (\mathbb{Z}/N\mathbb{Z})^2 \arrow[d] \\
& S
\end{tikzcd}
\Rightarrow \exists ! \, S \to \mathcal{M}_{N,p}$$

s.t. $E_1, E_2, \varphi, \alpha$ come by pullback

$\exists$ 2 maps

$$\begin{tikzcd}
& \mathcal{M}_{N,p} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_N & & \mathcal{M}_N
\end{tikzcd}$$

corresponding to $(\mathcal{E}_1, \alpha)$ & $(\mathcal{E}_2, \alpha \circ \varphi^{-1})$

$\pi_1 \; 1^*$

$\pi_2 = \begin{pmatrix} \pi_p & 0 \\ 0 & 1 \end{pmatrix}^*$

When he calculated it, he got $\begin{pmatrix} 1 & 0 \\ 0 & \pi_p \end{pmatrix}^*$. So normalisation may be wrong

$\pi_1^* \mathcal{L}_n = \mathcal{L}_n$

$\pi_2^* \mathcal{L}_n \xrightarrow[\begin{pmatrix} \pi_p & 0 \\ 0 & 1 \end{pmatrix}_*]{} \mathcal{L}_n$

π_1 (& π_2) are finite étale

$$\begin{tikzcd}
H^1(\mathcal{M}_N, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[r, "\pi_2^*"] & H^1(\mathcal{M}_{N,p}, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[d, equal, "(\pi_1 \text{ finite})"] \\
& H^1(\mathcal{M}_N, \pi_{1*} \pi_1^* \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[d, "\mathrm{tr}\ (\pi_1 \text{ finite \& flat or something})"] \\
& H^1(\mathcal{M}_N, \mathcal{L}_n(\mathbb{Z}_\ell))
\end{tikzcd}$$

? he doesn't mind which theory

sum of sthg or other

We understand tr /Q as sthg So we can check that the map overleaf is just

$$T_p : H^1(\mathcal{L}_n(\mathbb{Q}))^{U_N} \to H^1(\mathcal{L}_n(\mathbb{Q}))^{U_N}$$

$$\shortparallel$$

$$[U_N \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U_N]$$

The same goes for H_c & H_0

(The point is that sthg or-other (T_p?) makes sense in char p, whereas our other map (overleaf?) doesnt as we cant define the action of all the finite adèles)

Facts 1) $\mathcal{M}_N$ is defined (+ smooth quasi-proj) / spec $\mathbb{Z}[1/N]$ + all moduli-properties

2) $\mathcal{L}_n(\mathbb{Z}_\ell)$ plus all its properties works over spec $\mathbb{Z}[1/N\ell]$

$$\begin{tikzcd} & \mathcal{L}_n(\mathbb{Z}) / \mathcal{M}_N \arrow[d, "S"] & \\ \operatorname{spec} \mathbb{F}_p \arrow[r] & \operatorname{spec} \mathbb{Z}[1/N\ell] & \operatorname{spec} \mathbb{Q} \arrow[l] \end{tikzcd}$$

There's pullbacks in both cases

The fundamental fact is that the homologies of the pullbacks are often the same. This is useful in both directions (ie char 0 ⇝ char p & char p ⇝ char 0). There are comparison thms. He doesn't know which one to use right derived functor

$R^1 s_* \mathcal{L}_n(\mathbb{Z}_\ell)$ is a constructible ℓ-adic sheaf on spec $\mathbb{Z}[1/N\ell]$
+ on spec$(\mathbb{Z}_p)$.

V a finite $\mathbb{Z}_\ell$-module with a cts action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

W is a finite $\mathbb{Z}_\ell$-module with a cts action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$

$W \xleftarrow{\varphi} V$ s.t. φ compatible with Galois action

V is just H erm...

If this sheaf $R^i s_*(\mathcal{L})$ $s: X \to \mathrm{spec}\,\mathbb{Z}_p$ then $X \cong \mathbb{A}^1$

then $V = H^i(X \times \overline{\mathbb{Q}}_p, \mathcal{L})$ $W = H^i(X \times \overline{\mathbb{F}}_p, \mathcal{L})$

$H^1(\mathcal{M}_N \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell)) \to H^1(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))$, $p \nmid N\ell$
, compatible with actions of Galois gps.
In this case, it could well be an iso.

As RT understands it, $\mathcal{M}_{N\ell^r}$ / spec $\mathbb{Z}[1/N\ell]$ is smooth $\Rightarrow$ action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is unramified at p. (RT understands this pf)

Fact It appears that this map is iso.

The standard comparison thm is "iso when it's smooth & proper".
Our map is not proper

Deligne notes it's tamely ramified @ ∞ & refers to an article of Reynauld. RT understands neither the words "tamely ramified" nor the article.
Faltings compactifies & notes that smooth stable boundary lisse étale buzz-words hold so $\exists$ comparison thm, but Richard can't find it anywhere in SGA.

Def^n $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))/_{\overline{\mathbb{F}}_p} = \varinjlim_{(N,p)=1} H^1_p(X_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))$

$p \neq \ell$

$\prod_{r \neq p}^{\prime} \mathbb{Q}_r$ //

is an admissible $GL_2(\mathbb{A}^{\infty,p})$-module

cts module for $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$

& these commute

$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))/_{\mathbb{F}_p} \cong H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_p)}$

preserves $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ + $GL_2(\mathbb{A}^{\infty,p})$ actions

note: scheme $/\mathbb{F}_p$ don't get coh $/\mathbb{Q}$ - only $/\mathbb{Q}_p$

T_p acts here

(iso preserves T_p action once we've defined it see next!)

We can define $\mathcal{M}_{N,p}$ / spec $\mathbb{Z}[1/N]$

with elliptic curves $\mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2$, & $\mathcal{E}_1[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$

$$\begin{tikzcd} \mathcal{E}_1 \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{E}_2 \arrow[dl] \\ & \mathcal{M}_{N,p} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_1[N] \arrow[rr, "\sim"] \arrow[dr] & & (\mathbb{Z}/N\mathbb{Z})^2 \arrow[dl] \\ & \mathcal{M}_{N,p} & \end{tikzcd}$$

ker φ order p,

+ the obvious universal properties

As before, we have

$$\begin{tikzcd} & \mathcal{M}_{N,p} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_N & & \mathcal{M}_N \end{tikzcd}$$

$\mathcal{M}_{N,p}$ is regular, but no longer smooth

$\mathcal{M}_{N,p}$ is regular
π_1, π_2 are finite & flat, but not étale) crucial but

Thus we can define $T_p : H^1(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \circlearrowleft$ as before & this is compatible with the defn on $H^1(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_p)}$
(all you do is look in SGA until you find the facts that Deligne uses)

~~So $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))_{\overline{\mathbb{F}}_p} \xrightarrow{\cong} H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_p)}$~~
~~preserves T_p action too!~~

~~Also~~ Also for $H_{\mathbb{C}}, H_p$

Now ~~see~~ we can work in char p

$$\begin{tikzcd} X \arrow[d] \\ \mathbb{F}_p \end{tikzcd}$$

($\exists!$ $F : X \to X$
$F =$ id on top space
$F^* \ \mathcal{O}_X \to \mathcal{O}_X$
$f \mapsto f^p$

$(F \times 1)$ $X \times \overline{\mathbb{F}}_p \longrightarrow X \times \overline{\mathbb{F}}_p$

& $(1 \times \mathrm{Frob}^*)$ $X \times \overline{\mathbb{F}}_p \longrightarrow X \times \overline{\mathbb{F}}_p$

If ~~[illegible]~~ $\mathcal{F}/X$ is a lisse (locally free l-adic) sheaf

$\exists$ nat map $\mathcal{F} \to F^*\mathcal{F}$, & ~~[illegible]~~ thus we get 2 maps $\begin{matrix}(F\times 1)^* \\ (1\times \mathrm{Frob})^*\end{matrix}$: $H^i(X \times \overline{\mathbb{F}}_p, \mathcal{F}) \circlearrowleft$

↑ This last one gives the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$

It's easier here - the 'geometric Frobenius'

$(F \times 1)^* = F^*$ = geometric Frob

$(1 \times \text{Frob}^*)^*$ = arithmetic Frob

$(F \times 1)^* = (1 \times \text{Frob}^*)^{*\,-1}$

Chat about $\mathcal{F}$'s, & lisse sheaves

$$\pi_1(X) = \varprojlim_{\substack{Y/X \\ \text{finite} \\ \text{étale (= smooth unr.)} \\ \text{Galois}}} \text{Gal}(Y/X)$$

If $X/\text{Spec}\,\mathbb{C}$ even smooth proper, then

profinite cplt.

$$\pi_1(X) = \pi_1(X(\mathbb{C}))$$

lisse sheaves $\longleftrightarrow$ rep's of $\pi_1(X)$ on finite $\mathbb{Z}_\ell$-modules

may need to choose a base pt !!

In our case, $\mathcal{L}_n / \mathcal{M}_N$, $\mathcal{M}_{N\ell^\infty} \xrightarrow[\text{étale}]{} \mathcal{M}_N \rightsquigarrow$ n+1-dim' rep' (nth symm, ...)

$GL_2(\mathbb{Z}_\ell)$

Another way of looking at them

to give an étale sheaf on X is to give the following info.

$$\begin{tikzcd} V_{\text{finite}} \arrow[d, "\text{étale}"] \\ U \subseteq X \text{ open} \end{tikzcd}$$ $\mathcal{F}(V)$ ab gp, pullback of $U \hookleftarrow U'$ etc, ie def's compatible,

ab & of $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ locally cnsts

There's some notion of $\mathcal{F}$ being locally cst if $\mathcal{F}_n$ are sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-mods & $\exists$ maps $\mathcal{F}_n \to \mathcal{F}_{n-1}$

& these maps factor $\mathcal{F}_n \to \mathcal{F}_n \otimes \mathbb{Z}/\ell^{n}\mathbb{Z} \twoheadrightarrow \mathcal{F}_{n-1}$

$$\mathcal{F} =_{df} \varprojlim \mathcal{F}_n$$

Then $\mathcal{F}$ is lisse. lisse is stronger than constructible
He may have defined constructible.

It's not in Hartshorne.
There's a book by Milne on étale coh
Freitag
Also SGA 4½ Arcata talk by Déligne.

We've (next time) got to understand $\mathcal{M}_{N,p} \times \mathbb{F}_p$
- its not too bad.

Mon 3/2/92 Recall $\varinjlim H^1_p(\mathcal{M}_{N/\bar{\mathbb{Q}}}, \mathcal{L}_n(\mathbb{Z}_\ell)) = H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))$.

There was an action of

$$\{g \in GL_2(\mathbb{A}^\infty) \mid g_\ell \in M_{2\times 2}(\mathbb{Z}_\ell)\} \times Gal(\bar{\mathbb{Q}}/\mathbb{Q})$$

Also, $\varinjlim_{\substack{N \\ (N,p)=1}} H^1_p(\mathcal{M}_{N/\bar{\mathbb{F}}_p}, \mathcal{L}_n(\mathbb{Z}_\ell)) = H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))_{\bar{\mathbb{F}}_p}$, $p \neq \ell$.

There was an action of

$$\{g \in GL_2(\mathbb{A}^{\infty,p}) \mid g_\ell \text{ integral}\} \times Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p) \times \mathbb{Z}[T_p]$$

$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_p)}$ Action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is unramified at p ($p \neq \ell$)

$\wr\wr$

$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))_{\bar{\mathbb{F}}_p}$ Compatible with actions of
- Galois
- $GL_2(\mathbb{A}^{\infty,p})$
- T_p

Also, globally,

π-isotypical cpts

$$H^1_p(\mathcal{L}_n(\mathbb{Q})) = \bigoplus H^1_p(\mathcal{L}_n(\mathbb{Q}))(\pi)$$, π runs over irred. admiss. reps of $GL_2(\mathbb{A}^\infty)$ over $\mathbb{Q}$.

If $E_\pi = \mathrm{End}(\pi)$ then E_π is a number field

$$\rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{End}_{GL_2(\mathbb{A}^\infty)}\left(H^1_p(\mathcal{L}_n(\mathbb{Q}))(\pi) \otimes_{E_\pi} E_{\pi,\lambda}\right)$$

$$\cong$$

$GL_2(E_{\pi,\lambda})$ (?) (may be whole lot is $\cong GL_2(E_{\pi,\lambda})$)

$$\det \rho_\pi = (\chi^G_{\pi,\lambda}\, \chi_\ell)^{-1}$$

? $\mathrm{tr}\,\rho_\pi$?   In ptic, $\mathrm{tr}\,\rho_\pi(\mathrm{Frob}_p)$?

$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_p)}$ & $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))_{\bar{\mathbb{F}}_p}$ both have an action of Frob_p & the actions are compatible.

If $X/\mathbb{F}_p$ then recall $F : X \to X$ id on top spaces

$$\begin{array}{ccc} \mathcal{O}_X & \xleftarrow{F^*} & \mathcal{O}_X \\ f^p & \longmapsfrom & f \end{array}$$

$$\left.\begin{array}{l} F\times 1 \\ 1\times \mathrm{Frob}_p^* \end{array}\right\} : X \times \mathrm{Spec}\,\bar{\mathbb{F}}_p \to X \times \mathrm{Spec}\,\bar{\mathbb{F}}_p$$

⇝ Maps in cohomology:

$$H^i_{(?)}(X \times \mathrm{spec}\,\bar{\mathbb{F}}_p, \mathcal{F}) \circlearrowleft \begin{array}{l} (F\times 1)^* \\ (1\times \mathrm{Frob}_p^*)^* = \mathrm{Frob}_p \end{array}$$

Completely general Fact $(F\times 1)^* = \mathrm{Frob}_p^{-1}$.

pullback

$$\begin{tikzcd}
F^*\mathcal{E} \arrow[r] \arrow[d, "\text{Blimey}"'] & \mathcal{E} \text{ (universal ell. curve with level } N \text{ structure)} \arrow[d] \\
\mathcal{M}_N \times \mathbb{F}_p \arrow[r, "F"] & \mathcal{M}_N \times \mathbb{F}_p
\end{tikzcd}$$

& by some univ property, Φ exists:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, dashed, "\Phi"] \arrow[rr, bend left, "F"] \arrow[dr] & F^*\mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\sqcup"] & \mathcal{E} \arrow[d] \\
& \mathcal{M}_N \times \mathbb{F}_p \arrow[r, "F"] & \mathcal{M}_N \times \mathbb{F}_p
\end{tikzcd}$$

Fact: Φ an isogeny. Ker Φ has order p

Aside: If $E/\overline{\mathbb{F}}_p$ (+ level N structure)

Spec $\overline{\mathbb{F}}_p \to \mathcal{M}_N$

$$\begin{tikzcd}
E \arrow[r] \arrow[rr, bend left] \arrow[dr] & (\mathrm{Frob}_p^*)^* E \arrow[r] \arrow[d] & E \arrow[d] \\
& \mathrm{spec}\, \overline{\mathbb{F}}_p \arrow[r, "\mathrm{Frob}_p^*"] & \mathrm{Spec}\, \mathbb{F}_p
\end{tikzcd}$$

Φ specialises to an isogeny $E \to \mathrm{Frob}_p^{**} E$.

$y^2 = \overset{Q}{\cancel{p}}(x)$ $\qquad$ $y^2 = (\mathrm{Frob}_p Q)(x)$

Q ~~q~~ a cubic

& map sends $(x,y) \mapsto (\mathrm{Frob}_p x, \mathrm{Frob}_p y)$

Also $\exists$ dual isogeny ${}^t\Phi$: $F^*\mathcal{E} \to \mathcal{E}$, the dual of Φ

${}^t\Phi$ is an isogeny. Ker ${}^t\Phi$ has order p. $\Phi\,{}^t\Phi = p$

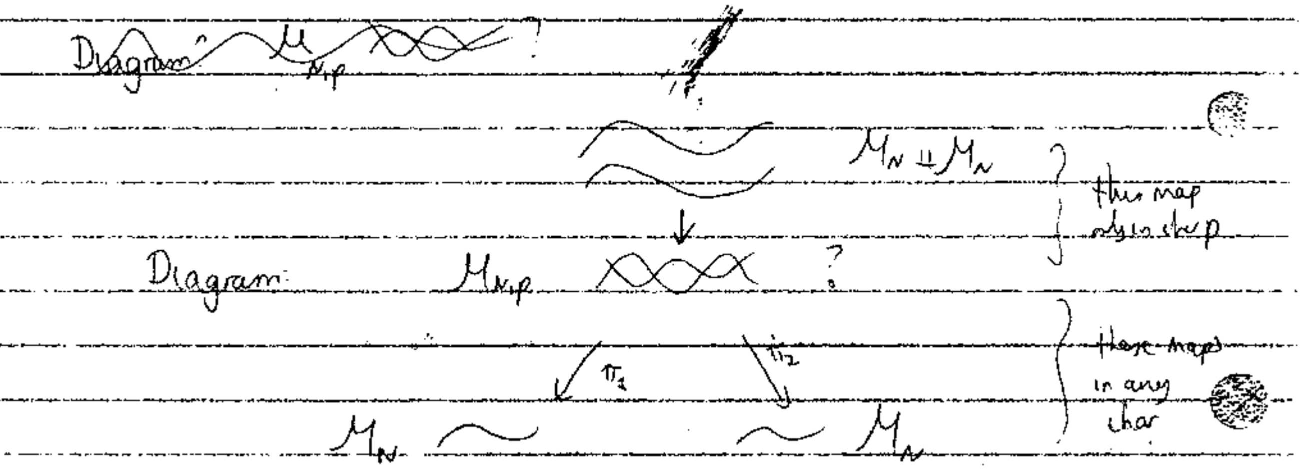
${}^t\Phi\,\Phi = p$.

We get

$$\begin{tikzcd} r: \mathcal{M}_N \times \mathbb{F}_p \arrow[rr, bend left, "r_1"] & \amalg\, \mathcal{M}_N \times \mathbb{F}_p \arrow[r] \arrow[r, bend right, "r_2"'] & \mathcal{M}_{N,p} \times \mathbb{F}_p. \end{tikzcd}$$

r_1 induced by $\mathcal{E} \xrightarrow{\Phi} F^*\mathcal{E} \,/_{\mathcal{M}_N \times \mathbb{F}_p}$

r_2 induced by $F^*\mathcal{E} \xrightarrow{{}^t\Phi} \mathcal{E} \,/_{\mathcal{M}_N \times \mathbb{F}_p}$

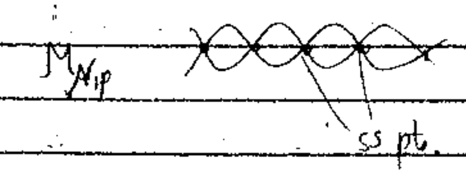


A finite number of the pts on $\mathcal{M}_N$ (or $\mathcal{M}_{N,p}$) correspond to supersingular elliptic curves (as ∃ only finitely many ss ell curves / $\overline{\mathbb{F}}_p$ in char p up to iso, or sthg)

Let $(\mathcal{M}_N \times \mathbb{F}_p)^\circ$, resp $(\mathcal{M}_{N,p} \times \mathbb{F}_p)^\circ$ denote the complement of these points. These are open & dense in $\mathcal{M}_N \times \mathbb{F}_p$ resp $\mathcal{M}_{N,p} \times \mathbb{F}_p$. Sometimes called the ordinary locus.

Fact: $r : (\mathcal{M}_N \times \mathbb{F}_p)^\circ \amalg (\mathcal{M}_N \times \mathbb{F}_p)^\circ \to (\mathcal{M}_{N,p} \times \mathbb{F}_p)^\circ$ is iso.

In ptic, $(\mathcal{M}_{N,p} \times \mathbb{F}_p)^\circ$ is smooth.



Not so difficult fact: just have to know that above an ordinary ell curve in char p ∃ just 2 isogenies whose kers have ord p

$$\begin{tikzcd}
 & \mathcal{M}_N \times \mathbb{F}_p \amalg \mathcal{M}_N \times \mathbb{F}_p \arrow[d, "r"] \arrow[ddl, bend right] \arrow[ddr, bend left, "\text{composition it's } F \amalg \mathrm{Id}"] & \\
 & \mathcal{M}_{N,p} \times \mathbb{F}_p \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_N \times \mathbb{F}_p & & \mathcal{M}_N \times \mathbb{F}_p
\end{tikzcd}$$

composition its id $\amalg_{r}$ F

thought of as a diag elt of $GL_2(\mathbb{A}^{\infty p})$ $GL_2(\mathbb{A}^{\infty p})$ ∴ action same as F_p

He thinks this isn't too difficult to see.

Pf. exercise.

eg 1st cpt of ~~RH composition~~ both compositions:

$\pi_1 \circ r_1$

$\pi_2 \circ r_1$

$$\begin{tikzcd}
 & \mathcal{E} \arrow[r, "\Phi"] \arrow[dl, "\text{pull back}"'] \arrow[dr] & F^*\mathcal{E} \arrow[d] \\
\mathcal{E} \arrow[dr] & \square & \mathcal{M}_N \times \mathbb{F}_p \arrow[dl, "\pi_1 \circ r_1"] \\
 & \mathcal{M}_N \times \mathbb{F}_p &
\end{tikzcd}$$

$\Rightarrow \pi_1 \circ r_1 = \mathrm{id}$ (it's a pullback, or sthg)

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\Phi"] \arrow[dd] & F^*\mathcal{E} \arrow[ddl] \arrow[dr] & \\
 & & \mathcal{E} \arrow[ddl] \\
\mathcal{M}_N \times \mathbb{F}_p \arrow[dr, "\pi_2 \circ r_1"] & & \\
 & \mathcal{M}_N \times \mathbb{F}_p &
\end{tikzcd}$$

$\Rightarrow \pi_2 \circ r_1 = F$

So what he's trying to say is that the diagram is easy. He may have got his π's mixed up. $\pi_1 \leftrightarrow \pi_2$.

Recall we had some funny old T_p s.

Lemma T_p can be calculated as

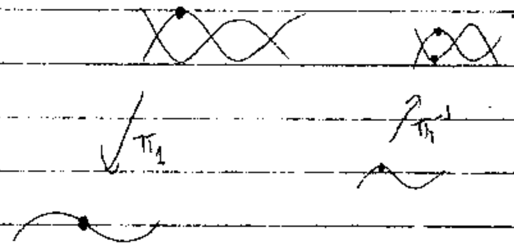
$$\begin{tikzcd}
H^1_p(\mathcal{M}_N\times\mathbb{F}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[r, "(F \amalg \mathrm{Id})^*"] & H^1_p(\mathcal{M}_N\times\overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2 \arrow[d, "\mathrm{tr}_{(\mathrm{Id} \amalg p^{-1}F)}"] \\
 & H^1_p(\mathcal{M}_N\times\overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))
\end{tikzcd}$$

Pf is in Deligne's Bourbaki talk

Idea: some map or other is nearly an iso.

In ptic, $z\in\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p$. $\pi_1^{-1}\pi_1(z)$ is a set with multiplicities

$\#\pi_1^{-1}\pi_1(z)$ = ~~[illegible]~~ $\#(\pi_1\circ F)^{-1}\pi_1 z.$



If ss then  has mult 2

□

RT doesn't really understand this.

Cor $T_p = F^* + p^{-1}\mathrm{tr}_F$ $\quad(\mathcal{M}_N\times\overline{\mathbb{F}}_p \xrightarrow{F} \mathcal{M}_N\times\overline{\mathbb{F}}_p)$

In ptic, $(X-F^*)(\underset{\substack{\text{called}\\ \text{"trace down } F\text{"}}}{X-p^{-1}\mathrm{tr}_F}) = X^2 - T_pX + p\,(\underset{\in GL_2(\mathbb{A}^{\infty,p})}{p^{-1}})$

& $\mathrm{Frob}_p^{-1} = F^*$ is a root of this poly.

So now we have a statement which doesn't involve any of the things which are only valid in char p.

So Frob_p^{-1} is a root of $X^2 - T_p X + p\langle p^{-1}\rangle$ on $H^1_p(\mathcal{L}_a(\mathbb{Z}_\ell))$

Hence if $\pi^{U_n} \neq (0)$ & $p \nmid N\ell$ then ρ_π is unr. at p, &

- $\det \rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1}) = (\chi^G_{\pi,\lambda}\chi_\ell)(\mathrm{Frob}_p)$

- ~~$X^2 - T_p$~~ $\rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1})^2 - \pi(\overset{\uparrow}{T_p})\rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1}) + \overset{\chi^G_{\pi,\lambda}\chi_\ell(\mathrm{Frob}_p)}{\cancel{\chi_\ell(\mathrm{Frob}_p)}}$

action under π of T_p ~~/ p~~

mult by p.

as $\chi^G_{\pi,\lambda}(\mathrm{Frob}_p) = \chi_\pi(\underset{GL_2(\mathbb{A}^{\infty,p})}{p^{-1}})$ with a lot of luck.

Let α, β be the roots of the char poly of $\rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1})$.

~~Then $\alpha\beta = \chi^G_{\pi,\lambda}\chi_\ell(\mathrm{Frob}_p^{-1})$~~

~~If $\alpha \neq \beta$ then $(X-\alpha)(X-\beta) = X^2 - \pi(T_p)X + (\chi^G_{\pi,\lambda}\chi_\ell)(\mathrm{Frob}_p)$~~

~~If $\alpha = \beta$~~ then $X^2 - \pi(T_p)X + (\chi^G_{\pi,\lambda}\chi_\ell)(\mathrm{Frob}_p)$

$$= (X-\alpha)\left(X - \frac{\det \rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1})}{\alpha}\right)$$

$$= (X-\alpha)(X-\beta) \text{ as } \alpha\beta = \det \rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1})$$

i.e. $\rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1})$ has char poly $X^2 - \underset{\substack{\uparrow \\ = \text{eigenvalue} \\ \text{of } T_p}}{\pi(T_p)}X + (\chi^G_{\pi,\lambda}\chi_\ell)(\mathrm{Frob}_p)$.

& in ptic $\mathrm{tr}\,\rho_{\pi,\lambda}(\mathrm{Frob}_p^{-1}) = \pi(T_p)$.

He'll try & write up a cleaner set of notes. He hasn't presented it so well. He found it v. difficult to understand

㊵ (NT)

$\rho_{\pi,\lambda}$

We've understood the local cpt at p of π_p unr

Next easiest case, Π_p special

Let's try & understand $\rho_{\pi,\lambda}\Big|_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$

confused notes

Π_p special $\Longleftrightarrow$ $\Pi_p^{U_1(p)}$ is 1-dim^l, $\Pi_p^{GL_2(\mathbb{Z}_p)} = (0)$

$\left.\begin{array}{l}\Pi_p = S(\chi, \chi|\cdot|_p) \\ \chi \text{ unr}\end{array}\right\} \Longleftrightarrow \Pi_p^{U_1(p)}$ is ~~1~~-dim^l, $\Pi_p^{GL_2(\mathbb{Z}_p)} = (0)$

→ deduce genl inc

Look at

$$H^1_p(\mathcal{L}_n(\mathbb{Z}_l))^{U_1(p)}$$
$$\circlearrowleft$$
$$Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$$

He now wants to say sthg about coh in char $p \mapsto$ char 0, which will cover what he said last time.

1) k perfect field : $\mathcal{F}$/ spec k ~~[illegible]~~ l-adic sheaf correspond to

$V / \mathbb{Z}_l$ finite(ly generated) module + action of $Gal(\bar{k}/k)$ on V.

via $V = H^0($~~[illegible]~~ spec $\bar{k}, \mathcal{F})$

2) $S =$ Spec $\mathbb{Z}_p$.

$$\begin{tikzcd} & \operatorname{Spec} \mathbb{Z}_p & \\ s = \operatorname{spec} \mathbb{F}_p \arrow[ur, "i"] & & \operatorname{spec} \mathbb{Q}_p = \eta \arrow[ul] \\ \bar{s} = \operatorname{spec} \bar{\mathbb{F}}_p \arrow[u] & & \operatorname{spec} \bar{\mathbb{Q}}_p = \bar{\eta} \arrow[u] \end{tikzcd}$$

let me go home;
let me go home,
I feel so exhausted;
let me go home

Hoist up the John B sail

$\mathcal{F}/S$ l-adic sheaf $\longleftrightarrow (F_{\bar{s}}, F_{\bar{\eta}}, \varphi)$ via

Ⓝⓣ (living hell) (nasty weekend!)

$i^*\mathcal{F} = F_{\bar s}$ (action of Gal$(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ on $F_{\bar s}$
action of Gal$(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $F_{\bar\eta}$

$j^*\mathcal{F} = F_{\bar\eta}$

$\varphi : F_{\bar s} \to F_{\bar\eta}^{I_{\mathbb{Q}_p}}$ which commutes with the action of Galois.
ie "Galois equivariant"

2 ways of getting φ:

Deligne: $\mathcal{F} \to j_*j^*\mathcal{F}$ corresponding to $j^*\mathcal{F} \xrightarrow{id} j^*\mathcal{F}$ (j_* adjoint to j^*)

$$\begin{tikzcd} i^*\mathcal{F} \arrow[r] \arrow[d] & i^*j_*j^*\mathcal{F}) \arrow[d] \\ \mathcal{F}_{\bar s} \arrow[r, "\varphi"] & F_{\bar\eta}^{I_{\mathbb{Q}_p}} \end{tikzcd}$$

$\rightsquigarrow$ according to Deligne

Other way ~~over~~ $H^0(\mathbb{Z}_p^{nr}, \mathcal{F})$ = stuff this.

Anyway, Delignes proved a thm saying $\exists$ bij between these 2 objects

$\mathcal{F}/S \longleftrightarrow (F_{\bar s}, F_{\bar\eta}, \varphi)$
l-adic sheaf

Facts: $j^*(F_{\bar s}, F_{\bar\eta}, \varphi) = F_{\bar\eta}$

$j_*(F) = (F^{I_{\mathbb{Q}_p}}, F, \text{incl})$

$i^*(F_{\bar s}, F_{\bar\eta}, \varphi) = F_{\bar s}$

$i_*(F) = (F, (0), \text{proj})$

$i^*j_*(F) = F^{I_{\mathbb{Q}_p}}$ (see def. of φ!!)

3) $$\begin{tikzcd} X \arrow[d, "f"] \\ S \end{tikzcd}$$

X a proper scheme over S. $\mathcal{F}/X$ lisse l-adic sheaf

Then ∃ exact seq. (coh of special fibre ~~r~~ stis)

$$\cdots \longrightarrow H^i(X_{\bar s}, \mathcal{F}) \xrightarrow{\circledast} H^i(X_{\bar\eta}, \mathcal{F}) \longrightarrow \mathbb{H}^i(X_{\bar s}, R\Phi_{\bar\eta}(\mathcal{F})) \rightarrow$$

special fibre generic fibre

complex of sheaves

where ~~eg~~ $Y_{\bar s} = X \times_S \bar s$, $X_{\bar\eta}$ $X \times_S \bar\eta$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_0 \arrow[r] \arrow[rr, bend left, "0"] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \cdots \arrow[r] & 0 \end{tikzcd}$$

Hypercoh $\mathbb{H}$ - resolution of n sheaves, diagonal maps: coh of diagonal H-ings

⊛ is the map induced by $R^i f_* \mathcal{F}/S$ l-adic sheaf: this corresponds to the triple

$$\begin{tikzcd} & R^i f_* \mathcal{F} \arrow[dl] \arrow[dr] & \\ H^i(X_{\bar s}, \mathcal{F}) \arrow[rr, "\rho_i"] & & H^i(Y_{\bar\eta}, \mathcal{F}) \end{tikzcd}$$

Help!

If $$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

$\bar f$ proper, $\mathcal{F}/X$, then

& $D \hookrightarrow \bar X$ a divisor with normal crossings (eg X a curve: D a set of pts)

st. ~~$\bar X \setminus$~~ D is the complement of X.

Assume $\exists$ nbd U of D & a finite surj cover $U' \xrightarrow{\pi} U$ s.t.

1) $\pi^{-1}(U \setminus D) \to U \setminus D$ is étale
2) π is tamely ramified along D
3) $\pi^{*}\mathcal{F}$ on $\pi^{-1}(U \setminus D)$ is free

then the previous result is still true.

RT doesn't understand 2). Deligne proves 2) is always satisfied by modular curves.

Recap

$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{M}_N / \operatorname{Spec} \mathbb{Z}[1/N] \end{tikzcd}$$

$\alpha : \mathcal{E}[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\times}$

univ property

$n \in \mathbb{Z}_{\geq 0}$, $\mathcal{L}_n(\mathbb{Z}_\ell) / \mathcal{M}_N \times \operatorname{spec} \mathbb{Z}[1/N\ell]$

lisse ℓ-adic sheaf

$$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)) = \varinjlim_N \operatorname{Im}\left(H^1_c(\mathcal{M}_N \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell)) \to H^1(\mathcal{M}_N \times \overline{\mathbb{Q}}, \mathcal{L}_n)\right)$$

~~Then $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))$~~

$$H^1_p(\mathcal{L}_n(\mathbb{Q})) = \varinjlim_N \operatorname{Im}\left(H^1_c(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q})) \to H^1(\mathcal{M}_N(\mathbb{C}), \mathcal{L}_n(\mathbb{Q}))\right)$$

$$H^1_p(\mathcal{L}_n(\mathbb{Q})) \otimes \mathbb{Q}_\ell \cong H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

$H^1_p(\mathcal{L}_n(\mathbb{Q}))$ is an admissible $GL_2(\mathbb{A}^\infty)$-module

$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))$ is an admiss module for $\{ g \in GL_2(\mathbb{A}^\infty) \mid g_\ell \in M_{2\ell}(\mathbb{Z}_\ell) \}$

If **N**≥ 3 then $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_N} = H^1_p(\mathcal{M}_N \times \bar{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$ etc

$$H^1_p(\mathcal{L}_n(\mathbb{Q})) = \bigoplus_{\substack{\text{irred} \\ \text{admiss} \\ \text{rep's } \pi \\ \text{of } GL_2(\mathbb{A}^\infty) \\ \text{over } \mathbb{Q}}} \pi \otimes_{E_\pi} W_\pi$$

where $E_\pi = \mathrm{End}_{GL_2(\mathbb{A}^\infty)}(\pi)$ is a no. field

W_π is 2 dim^l / E_π

$$S_{n+2} \cong \bigoplus_{\substack{\text{same} \\ \pi}} \bigoplus_{\tau : E_\pi \to \mathbb{C}} \overbrace{(\pi \otimes_{E_\pi, \tau} \mathbb{C})}^{\text{irred admiss } / \mathbb{C}}$$

he wrote $\otimes$

$$H^1_p(\mathcal{L}_n(\mathbb{Q}_\ell)) = \bigoplus_{\substack{\text{same} \\ \pi}} \bigoplus_{\substack{\text{primes } \lambda \\ \text{of } E_\pi \\ \lambda | \ell}} \pi \otimes_{E_\pi} W_{\pi,\lambda} \longleftarrow \text{completion of } W_\pi \text{ @ } \lambda$$

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))$. This commutes with the action of $GL_2(\mathbb{A}^\infty)$. So we get $\rho_{\pi,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(E_{\pi,\lambda})$

We know $\det \rho_{\pi,\lambda} = (\chi^{q}_{\pi,\lambda} \chi_\ell)^{-1}$

$$\rho_{\pi,\lambda}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We're now going to examine $\rho_{\pi,\lambda}\big|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ a little more
Again assume $l \neq p$.

Put $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)/\bar{\mathbb{F}}_p) = \varinjlim_{\substack{N \\ (N,p)=1}} \mathrm{Im}(H^1_c(\mathcal{M}_N \times \bar{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to H^1(\text{ditto}))$

Then $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)/\bar{\mathbb{F}}_p)$ has natural actions of
- $GL_2(\mathbb{A}^{\infty,p})$
- $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$
- T_p

$$\begin{tikzcd} E_1 \arrow[r, "\varphi"] \arrow[dr] & E_2 \arrow[d] \\ & \mathcal{M}_{N,p} \end{tikzcd}$$

$\deg \varphi = p$, $\alpha : E_1[N] \to (\mathbb{Z}/N\mathbb{Z})^2$

$\mathcal{M}_{N,p}$ / Spec $\mathbb{Z}[1/N]$ universal

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\psi"] \arrow[dr] & \mathcal{E}_2 \arrow[d] & \\
\text{regular} \arrow[r] & \mathcal{M}_{N,p} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_N & & \mathcal{M}_N
\end{tikzcd}$$

finite & flat

$$\begin{tikzcd}
T_p \, H^1_p(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[r, "\pi_2^*"] & H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[r, "tr_{\pi_1}"] & H^1_p(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))
\end{tikzcd}$$

$$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_\ell)} \cong H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)/\overline{\mathbb{F}}_p)$$

$$GL_2(\mathbb{A}^{\infty,p}) \hookrightarrow GL_2(\mathbb{A}^{\infty,p})$$

$$\underset{\text{unramified}}{Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \twoheadrightarrow Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

$$\left[GL_2(\mathbb{Z}_p)\begin{pmatrix}1&0\\0&\pi_p\end{pmatrix}GL_2(\mathbb{Z}_p)\right] \longleftrightarrow T_p$$

π occurring in $H^1_p(\mathcal{L}_n(\mathbb{Q}))$

Π_p unramified, $p \neq \ell \Rightarrow \rho_{\pi,\lambda}$ is unramified at p

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\Phi"] \arrow[d] \arrow[rr, bend left, "F"] & F^*\mathcal{E} \arrow[r] \arrow[dl] & \mathcal{E} \arrow[d] \\
\mathcal{M}_N \times \mathbb{F}_p \arrow[rr, "F"] & & \mathcal{M}_N \times \mathbb{F}_p
\end{tikzcd}$$

Φ an isogeny deg p

$$\begin{tikzcd}
F^*\mathcal{E} \arrow[r, "{}^t\Phi"] \arrow[dr] & F^*\mathcal{E} \arrow[d] \\
& \mathcal{M}_N \times \mathbb{F}_p
\end{tikzcd}$$

$\gamma_p = M_2 \in GL_2(\hat{\mathbb{Z}}^p)$

$$\begin{tikzcd}
& \mathcal{M}_N \times \mathbb{F}_p \amalg \mathcal{M}_N \times \mathbb{F}_p \arrow[dl, bend right, "\mathrm{id} \amalg \gamma_p^* F^\vee"'] \arrow[d, "r"] \arrow[dr, bend left, "F^* \amalg \mathrm{id}"] & \\
& \mathcal{M}_{N,p} \times \mathbb{F}_p \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_N \times \mathbb{F}_p & & \mathcal{M}_N \times \mathbb{F}_p
\end{tikzcd}$$

r is an iso on ordinary locus (dense)
r is $2 \Rightarrow 1$ on the remaining pts
r gives the normalisation of $\mathcal{M}_{N,p} \times \mathbb{F}_p$. at the bad points the 2 copies of $\mathcal{M}_N \times \mathbb{F}_p$ cut transversally

(All this depends on $N \geq 3$)

Note: he may have put γ_p^{+1} before. $^{-1}$ is right.

$\Rightarrow T_p = F^* + \gamma_p^{-1} \mathrm{tr}_F$

$\Rightarrow (\mathrm{Frob}_p)^{-1}$ satisfies $X^2 - T_p X + (\chi^G_{\pi,\lambda} \chi_\ell)(\mathrm{Frob}_p)$

on $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)/\mathbb{F}_p)$

& hence on $W_{\pi,\lambda}$ if π_p unr.

$\Rightarrow \rho_{\pi,\lambda}(\mathrm{Frob}_p)^{-1}$ has char poly $X^2 - \pi_p(T_p)X + (\chi^G_{\pi,\lambda} \chi_\ell)(\mathrm{Frob}_p)$

$\mathrm{tr}\, \rho_{\pi,\lambda}(\mathrm{Frob}_p)^{-1} = \pi_p(T_p)$

This is about where we'd got to

That was about where we'd got to

Next we'll try & understand $\rho_{\pi,\lambda}\Big|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ when $\pi = S(\chi, \chi|\cdot|_p)$,

χ unramified

$\updownarrow$

understanding the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_0(p)}$

$U_0(p) \subseteq \mathrm{GL}$[illegible]$(\mathbb{Z}_p)$

$\uparrow$ upper Δ mod p.

$\leftrightarrow$ action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))$

hypercoh

$$\cdots \; H^i_{(c)}(\underset{\text{special fibre}}{\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p}, \mathcal{L}_n(\mathbb{Z}_\ell)) \longrightarrow H^i_{(c)}(\mathcal{M}_{N,p} \times \overline{\mathbb{Q}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \rightarrow \mathbb{H}^i_{(c)}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R\Phi\,\mathcal{L}_n(\mathbb{Z}_\ell)) \rightarrow \cdots$$

is true if $\mathcal{M}_{N,p} / \mathbb{Z}[1/N]$ is proper, which it's not.

However, it's also true if $\mathcal{M}_{N,p} \hookrightarrow \underset{\text{compactification}}{\mathcal{M}^*_{N,p}} / \mathbb{Z}[1/N]$ is proper

& 1) $\mathcal{M}^*_{N,p} - \mathcal{M}_{N,p}$ is a divisor with normal ($\perp$) crossings

2) $\mathcal{L}_n(\mathbb{Z}_\ell)$ is locally cst in a nbd of the divisor & tamely ramified at the divisor.

Deligne has proved 2). So we're OK.

Def of tamely ramified:

$\forall r.$ $\mathcal{L}_n(\mathbb{Z}/r\mathbb{Z})$, $U \supseteq D$ open, U' finite,

$$\begin{tikzcd} U' \setminus \pi^* D \arrow[d, "\text{étale}"] & U' \arrow[d, "\pi"'] \\ U - D & U \end{tikzcd}$$

$\pi^*\mathcal{L}_n(\mathbb{Z}/\ell^r\mathbb{Z})$ trivial

& ramification along D has order prime to p
(in our case, ram is ℓ-power)

In our case we're quite lucky 'cos the singularities of $\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p$ (red^n r) are quadratic (X) & non-degenerate (normal crossings) & Deligne has calculated ~~[illegible]~~ for quadratic non-deg singularities in SGA 7 II (any dimension) $R\Phi\mathcal{L}_n(\mathbb{Z}_\ell)$
vol (in Picard Lefschetz)

(local rings $\mathbb{Z}_p[[X_1 .. X_{d+1}]] / (Q(x) + \text{higher})$:
quad form

Fact (Deligne):

$$R\Phi^i\mathcal{L}_n(\mathbb{Z}_\ell) = \begin{cases} 0 & \text{if } i \neq 1 \quad (=\text{dimension}) \\ \bigoplus\limits_{x\in\Sigma} (R\Phi^1\mathbb{Z}_\ell)_x \otimes \mathcal{L}_n(\mathbb{Z}_\ell)_x \end{cases}$$

!

Σ = singularities of $\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p$

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & H^1_c(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[r] \arrow[dd] & H^1_c(\mathcal{M}_{N,p}\times\overline{\mathbb{Q}}_p, \mathcal{L}_n) \arrow[r] \arrow[dd] & H^0_c(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, R^1\Phi\mathcal{L}_n) \arrow[r] \arrow[dd, equal] & {} \\ \cdots \arrow[r] & H^2_c(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[r] & H^2_c(\mathcal{M}_{N,p}\times\overline{\mathbb{Q}}_p, \mathcal{L}_n) \arrow[r] & 0 & \\ \& \ 0 \arrow[r] & H^1(\mathcal{M}_{N,p})\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[r] & H^1(\mathcal{M}_{N,p}\times\overline{\mathbb{Q}}_p, \mathcal{L}_n) \arrow[r] & H^0(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, R^1\Phi\mathcal{L}) \arrow[r] \arrow[d, equal] & 0 \\ & & & \bigoplus\limits_{x\in\Sigma}(R^1\Phi\mathbb{Z}_\ell)_x\otimes\mathcal{L}_n(\mathbb{Z}_\ell)_x & \end{tikzcd}$$

So $0 \to H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n) \to H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{Q}}_p, \mathcal{L}_n)$

$$\to \bigoplus_{x \in \Sigma} (R^1 \Phi \mathbb{Z}_\ell)_x \otimes \mathcal{L}_n(\mathbb{Z}_\ell)_x \to H^2_c(-) \to H^2_c(-) \to 0$$

$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts on each of these objects, all maps preserve this action.

$I_{\mathbb{Q}_p}$ acts via $\sigma \mapsto 1 + \mathrm{var}(\sigma)$ where

$$\mathrm{var}(\sigma) : \bigoplus_{x \in \Sigma} (R^1 \Phi \mathbb{Z}_\ell)_x \otimes \mathcal{L}_n(\mathbb{Z}_\ell)_x \to H^1_c(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n)$$

$$\|$$

$\sum_{x \in \Sigma} \mathrm{var}_x(\sigma) \otimes id$ where $\mathrm{var}_x(\sigma) : (R^1 \Phi \mathbb{Z}_\ell)_x \to H^1_c(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n)$

Lets think about the H^1_p of the special fibre. Even this isn't such a simple object.

$$\begin{tikzcd} \text{pull it back to } \mathcal{L}_n(\mathbb{Z}_\ell) & \mathcal{M}_N \times \overline{\mathbb{F}}_p \sqcup \mathcal{M}_N \times \overline{\mathbb{F}}_p \arrow[d, "r"] \\ \mathcal{L}_n(\mathbb{Z}_\ell) \arrow[u] & \mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p \end{tikzcd}$$

$0 \to \mathcal{L}_n(\mathbb{Z}_\ell) \to r_* r^* \mathcal{L}_n(\mathbb{Z}_\ell)) \to \mathcal{G}_n \to 0$ where $\mathcal{G}_n$ is supported on the singular set

$$H^i_?(\mathcal{M}_N \times \overline{\mathbb{F}}_p \sqcup \mathcal{M}_N \times \overline{\mathbb{F}}_p, r^* \mathcal{L}_n(\mathbb{Z}_\ell)) \cong H^i_?(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2$$

$$\wr\|$$

$$H^i_?(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, r_* r^* \mathcal{L}_n(\mathbb{Z}_\ell))$$

Get another complicated long exact seq.

cpt suppt vanishes

$$0 \to H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to H^0(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2 \to \bigoplus_{x\in\Sigma} \mathcal{G}_x \to H^1(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to$$

$$\to H^1(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2 \to 0$$

& similarly with cpct support : get

$$0 \to \bigoplus_{x\in\Sigma} \mathcal{G}_x \to H^1_c(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to H^1_c(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2 \to 0$$

∴ parabolic : get

$$0 \to \underbrace{H^0(-) \to H^0(-)}_{0 \text{ if } n>0} \to \bigoplus_{x\in\Sigma} \mathcal{G}_x \to H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to$$

equal

$$\to H^1_p(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2 \to 0$$

Also $0 \to H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{Q}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to$

$$\to \bigoplus_{x\in\Sigma} (R^1\Phi\,\mathbb{Z}_\ell)_x \otimes \mathcal{L}_n(\mathbb{Z}_\ell), \to$$

$$\to H^2_c(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \to \underbrace{H^2_c(\mathcal{M}_{N,p} \times \overline{\mathbb{Q}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))}_{0 \text{ if } n\geq 0} \to 0$$

Here's some sort of duality between the H^0_c & the H^2_c s. He'll explain this next wk when he's understood it.

$\Sigma_N = \{$ supersingular pts on $\mathcal{M}_N \times \overline{\mathbb{F}}_p \}$

$\leftrightarrow \{$ iso classes of pairs $(E,\alpha) \mid E/\overline{\mathbb{F}}_p$ is a ss ell. curve, & α is a level N structure $\}$

Set $TE = \varprojlim_N E[N][\overline{\mathbb{F}}_p] \underset{\substack{\uparrow \\ \text{non-canonical}}}{\cong} (\hat{\mathbb{Z}}^p)^2$, $\hat{\mathbb{Z}}^p = \prod_{q \neq p} \mathbb{Z}_q$

(Tate module)

$VE = TE \otimes_{\hat{\mathbb{Z}}^p} \mathbb{A}^{\infty,p}$

<u>Lemma</u> $\exists$ bij $\Sigma_N \leftrightarrow \left\{ (E,A) \,\middle|\, \begin{array}{l} E/\overline{\mathbb{F}}_p \text{ n.b. c.c.} \\ A : VE \xrightarrow{\sim} \underset{\uparrow \text{ length 2 row vector}}{(\mathbb{A}^{\infty,p})^2} \end{array} \right\} / \sim$

where $(E,A) \sim (E',A')$ if $\exists \varphi : E \to E'$ isogeny, $(\deg \varphi, p) = 1$,

$A' = u \cdot A \cdot \varphi^{-1}$, some $u \in U_N^p$

Pf $(E_0, \alpha) \longleftarrow\!\!\shortmid (E,A)$

$E_0 \hookrightarrow A^{-1}(\hat{\mathbb{Z}}^p)^2$, $\alpha : E_0[N] \xrightarrow{A} N^{-1}(\hat{\mathbb{Z}}^p)^2 / (\hat{\mathbb{Z}}^p)^2 \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$.

<u>exercise</u> check it's bij. $\square$

<u>Rks</u> 1) $g^*(E,A) = (E, g \circ A)$, $g \in GL_2(\mathbb{A}^{\infty,p})$

warning Careful: g acts on it on row vectors

2) If $E/\mathbb{F}_p$ then $F^*(E,A) = (E, A \circ \text{Frob}_p)$, $\text{Frob}_p \in \text{End}_{\mathbb{F}_p}(E)$

There seems to be some confusion about this lemma: RT thinks what he's written might be false.

<u>Lemma</u> 1) If $E_1, E_2 / \overline{\mathbb{F}}_p$ are 2 ss ell curves, $\exists$ isogeny of degree prime to p, $\varphi : E_1 \to E_2$.

2) $\exists\, E/\mathbb{F}_p$ ss ell curve

<u>Pf</u> 2) ✓ using Hyundai-Tate theory (<u>Honda</u>-Tate really, fool)

1) all ss ell curves $/\overline{\mathbb{F}}_p$ are isogenous. Possibly $\square$.

So $\exists \varphi_1 : E_1 \to E_2$ isog. Then $\varphi_1 = \varphi \Phi^r$, $\Phi = \text{Frob.}$, φ deg prime to p.

WLOG $E_1/\mathbb{F}_p$. Then $\Phi = \mathrm{Frob} \in \mathrm{End}_{\mathbb{F}_p}(E_1)$ & $\varphi: E_1 \to E_2$. $\square$

Cor $E_1/\mathbb{F}_p$ ss ec. Then $\Sigma_N \leftrightarrow \{(E_1, A) \mid A: V E_1 \xrightarrow{\sim} (\mathbb{A}^{\infty,p})^2\}/\sim$

$(E_1, A) \sim (E_1, A') \Leftrightarrow \exists\, \varphi \in \mathrm{End}^\circ_{\mathbb{F}_p}(E_1)$,

$(\deg \varphi, p) = 1$, $u \in U_N$,
s.t. $A' = u \circ A \circ \varphi^{-1}$.

Fix one such A_1. We get...

$\Big[$ Fix $D = \mathrm{End}^\circ_{\mathbb{F}_p}(E_1) = \mathrm{End}_{\mathbb{F}_p}(E_1) \otimes_{\mathbb{Z}} \mathbb{Q}$, $D/\mathbb{Q}$ quat. alg.

$S(D) = \{\infty, p\}$

set $D^p = D \cap \mathcal{O}_{D_p}$ = things in D with no peas in denominator $\Big]$

$$\Sigma_N \longleftrightarrow (D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}) / U_N$$

Moreover the action of $GL_2(\mathbb{A}^{\infty,p})$ is gl.

$h^*: [g] \mapsto [gh]$

& $F^*[g] = [\underset{\underset{D^\times}{\cap}}{\mathrm{Frob}_p} g]$ if he's got it right.

$\Big[$ glimmer of hope: $n=0$. $\bigoplus_{x \in \mathcal{G}} g_x$ "=" $\mathrm{Map}(\Sigma_N, \mathbb{Z}_\ell) \simeq S_2^D(\mathbb{Z}_\ell)^{U_N}$

$S_2(U_N) \hookrightarrow S_2(U_0(p)) \cap U_N$

Lecture 17
9/3/92

Recall $\overset{\text{s.s. pts}}{\Sigma_N} \hookrightarrow \mathcal{M}_N \times \overline{\mathbb{F}}_p$

$\underset{\text{ssig pts}}{\Sigma^*_N} \longrightarrow \mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p$

Fix $E_1/\overline{\mathbb{F}}_p$ a ss ell. curve

Set $\mathcal{D} = \mathrm{End}^\circ_{\overline{\mathbb{F}}_p}(E_1)$ $\mathcal{D}/\mathbb{Q}$ quat. alg. $S(\mathcal{D}) = \{p, \infty\}$

$\pi \in \mathcal{D}$, $\pi^2 = -p$, π represents Frobenius $E_1 \to E_1$

$\mathcal{D}^p =$ elts of $\mathcal{D}$ integral at p $= \mathcal{D} \cap \underset{\overset{\uparrow}{(!)}}{\mathcal{O}_{\mathcal{D}_p}}$ Fix $\alpha_1 : E_1[N](\overline{\mathbb{F}}_p) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$

$$\Sigma_N \cong (\mathcal{D}^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}) / U^p_N$$

$$\Sigma = \varprojlim_N \Sigma_N \quad \begin{cases} g \in GL_2(\mathbb{A}^{\infty,p}) & g^*(h) = hg \\ F^*(h) = \pi^{-1} h & \end{cases}$$

We want to understand Σ^*_N really. But

$$\begin{tikzcd}
& \Sigma_N \amalg \Sigma_N \arrow[dl, bend right, "\mathrm{id} \amalg F^*\gamma_p^*"'] \arrow[d, "r"] \arrow[dr, bend left] & \\
& \Sigma^*_N \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\Sigma_N & & \Sigma_N
\end{tikzcd}$$

$\gamma_p = p 1_2 \in GL_2(\mathbb{A}^{\infty,p})$

Fix $\Sigma^*_N \xrightarrow[\sim]{\pi_1} \Sigma_N$. $\pi_0(\mathcal{M}_N \times \overline{\mathbb{F}}_p) \overset{\text{as before}}{\cong} \left\{ \begin{matrix} \text{prim} \\ N^{\text{th}} \text{ roots} \\ \text{of unity} \end{matrix} \right\}$

(check connected)

$$\mathcal{M}_N \times \overline{\mathbb{F}}_p = \coprod_J \mathcal{M}_{N,J} \qquad \begin{tikzcd} \Lambda^2 \mathcal{E}[N] \arrow[r, "\Lambda^2\alpha", "\sim"'] \arrow[d, phantom, "\cong" rotate] & \Lambda^2(\mathbb{Z}/N\mathbb{Z})^2 \arrow[d, phantom, "\cong"] \\ \mu_N & \mathbb{Z}/N\mathbb{Z} \\ J & 1 \arrow[l, mapsto] \end{tikzcd}$$

Choice of (E_1, α_1) gives J_1 (= image of 1 in this case)

$$\pi_0(\mathcal{M}_N \times \overline{\mathbb{F}}_p) \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

$$\mathcal{M}_{N,J_1^a} \longleftarrow\!\shortmid\; a \qquad \cong$$

$$\underbrace{(\mathbb{Q}^p)^\times}_{\text{elts of } \mathbb{Q} \text{ integral at } p} \backslash \mathbb{A}^{\infty,p\,\times} / \det U_N^p$$

Then $\Sigma_N \twoheadrightarrow \pi_0(\mathcal{M}_N \times \overline{\mathbb{F}}_p)$

$(E,\alpha) \mapsto$ cpt on which it lies

$$(D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p})^\times / U_N^p \xrightarrow{\det^{-1}} (\mathbb{Q}^p)^\times \backslash (\mathbb{A}^{\infty,p})^\times / \det^* U_N^p$$

(ex check this)

$$\pi_0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p) \xrightarrow[\sim]{\pi_1} \pi_0(\mathcal{M}_N \times \overline{\mathbb{F}}_p)$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_n \arrow[r] & r_* r^* \mathcal{L}_n \arrow[r] & \mathcal{G}_n \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{L}_n|_{\Sigma_N^*} \arrow[r] \arrow[u, hook] & r_* r^* \mathcal{L}_n|_{\Sigma_N^*} \arrow[r] \arrow[u, hook] & \mathcal{G}_n \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

possibly same as $\mathcal{L}_n$ but (unembedded) on Σ_N^*. Possibly $\Sigma_N^* \hookrightarrow \mathcal{M}_N \times \overline{\mathbb{F}}_p$ $\mathcal{L}$ its $\pi_*\pi^*\mathcal{L}$. He wouldn't swear to that.

$$H^0(\mathcal{G}_n) := \varinjlim_N H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{G}_n)$$

as sthg is concentrated in pts. higher cohomologies vanish.

So we get an exact seq

$$0 \to Map(\Sigma_N^*, S^n(\mathbb{Z}_\ell^2)) \xrightarrow{r^*} Map(\Sigma_N, S^n(\mathbb{Z}_\ell^2))^2 \longrightarrow H_0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{G}_n) \to 0$$

Take direct limits / N & do sthg else too & identify Σ_N^* with Σ_N via π_1

$$0 \to \overset{\text{locally cst}}{Map}((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), S^n(\mathbb{Z}_\ell^2)) \xrightarrow[\text{(see margin)}]{id^* \oplus (\delta_p^* F^*)^*} Map((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), S^n(\mathbb{Z}_\ell^2))^2 \to H^0(\mathcal{G}_n) \to 0$$

$\Sigma_N \amalg \Sigma_n$

$$\begin{tikzcd} \Sigma_N^* \arrow[loop left, "F"] \arrow[d, "\pi_1"] \\ \Sigma_N \end{tikzcd}$$

ie $f \longmapsto f, (f(-\pi\_))$

Moreover we can keep track of $GL_2(\mathbb{A}^{\infty,p})$ actions: g acts by $g_*(f) = g_1 f(-g)$

On first Map, $F_*(f) = f(\pi^{-1}\_)$

On second $F_*(f_1, f_2) = f_2(\gamma_p^{-1}\_), f_1(\_))$

So we have $H^0(\mathcal{G}_n)$

In case $n=0$ we're rather more interested in something else, however, the quotient.

$$H^0(\mathcal{G}_n)^{triv} = \begin{cases} (0) & n>0 \\ Im((\varinjlim_N H^0(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathbb{Z}_\ell)) \to H^0(\mathcal{G}_0)) & n=0 \end{cases}$$

adèle

idèle

Define $H^0(\mathcal{G}_n)^- = H^0(\mathcal{G}_n) \big/ H^0(\mathcal{G}_n)^{triv}$

So it remains to describe $H^0(\mathcal{G}_n)^{triv}$ in terms of the adèlic setting.

~~$Map((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p})$~~

$$\begin{tikzcd} & & & H^1(\mathcal{G}_0) \arrow[r] & 0 \\ 0 \arrow[r] & Map((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), \mathbb{Z}_\ell) \arrow[r] & Map((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), \mathbb{Z}_\ell)^2 \arrow[r] & H^0(\mathcal{G}_0) \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & Map((\mathbb{Q}^p)^\times \backslash (\mathbb{A}^{\infty,p})^\times, \mathbb{Z}_\ell) \arrow[r] \arrow[u, hook, "(\det^{-1})^*"] \arrow[d, phantom, "\parallel"] & Map((\mathbb{Q}^p)^\times \backslash (\mathbb{A}^{\infty,p})^\times, \mathbb{Z}_\ell)^2 \arrow[u, hook, "(\det^{-1})^*"] \arrow[ur] \arrow[d, phantom, "\parallel"] & & \\ & \varinjlim_N H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathbb{Z}_\ell) & \varinjlim_N H^0(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathbb{Z}_\ell)^2 & & \end{tikzcd}$$

ex: check commutes

(Hint $0 \to \mathcal{L}_n \to r_* r^* \mathcal{L}_n \to \mathcal{G}_n \to 0$

$0 \to \mathcal{L}_n \to r_* r^* \mathcal{L}_{n, \Sigma_n} \to \mathcal{G}_n \to 0$

$n=0$)

Lemma

1) $H^0(\mathcal{G}_n) \cong \mathrm{Map}\left((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), S^n(\mathbb{Z}_\ell^2)\right)$

$g \in GL_2(\mathbb{A}^{\infty,p})$ $\qquad g_*(f) = g_\ell f(\cdot g)$

$F^*(f) = -f(\pi^{-1}\_)$

$$H^0(\mathcal{G}_n)^{\mathrm{triv}} = \begin{cases} (0) & n>0 \\ \{f : (D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}) \xrightarrow{\cancel{\det}} (\mathbb{Q}^p)^\times \backslash \mathbb{A}^{\infty p \times} \to \mathbb{Z}_\ell\} & n=0 \end{cases}$$

Recall from last time the reason for doing this.

2) $0 \to H^0(\mathcal{G}_n)^{\Gamma} \to \underbrace{H^1(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \xrightarrow{r^*} H^1(\mathcal{M}_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))^2}_{} \to 0$

So we now have fairly explicitly described the cohomology of the singular reduction.

Next understand its rel^n to coh in char 0 & vanishing cycles (as in SGA 7)

∃ complex object which specialises to $H^0(\mathcal{G}_n)^\Gamma$ ~ stly ← in SGA 7

Recall $0 \to \mathcal{L}_n \to c_* r^* \mathcal{L}_n \to \mathcal{G}_n \to 0$

Take coh with support on Σ_N^*. (see over for pf) (Conrad's remarks)

$(0 \to)\ H^0_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{G}_n) \to H^1_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n)\ (\to 0)$

as they're lisse

as $\mathcal{L}_n$ Lisse M_N smooth

RT doesn't understand.

He wants $H^1_{\Sigma_N^*}(M_N \times \overline{\mathbb{F}}_p, \mathcal{L}_n) = 0$.

So it's so.

Moreover it fits in with exact sequences

$$\begin{tikzcd}
H^0_{\Sigma_N^*}(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{G}_n) \arrow[r, "\sim"] \arrow[d, equal] & H^1_{\Sigma_N^*}(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \\
H^0(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \overset{*}{\mathcal{G}_n}) &
\end{tikzcd}$$

so we have a new description of this

vanishes as above

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & 0 \arrow[r] & H^0(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[r] \arrow[ddr] & H^0(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p - \Sigma_N^*, \mathcal{L}_n) \arrow[r] \arrow[dd, "\wr"] & H^1_{\Sigma_N^*}(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[d, bend left] & \\
 & & & & H^1(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[r] \arrow[u] \arrow[d, "\wr"] & H^1(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p - \Sigma_N^*, \mathcal{L}_n) \\
 & & & H^0(\mathcal{M}_N\times\overline{\mathbb{F}}_p - \Sigma_N \amalg \mathcal{M}_N\times\overline{\mathbb{F}}_p - \Sigma_N, \mathcal{L}_n) \arrow[r] & H^0(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{G}_n) \arrow[d, bend left] & \\
 & & & & H^1(\mathcal{M}_{N,p}\times\overline{\mathbb{F}}_p, \mathcal{L}_n) \arrow[r] & H^1(\mathcal{M}_N\times\overline{\mathbb{F}}_p \amalg \mathcal{M}_N\times\overline{\mathbb{F}}_p, \mathcal{L}_n)
\end{tikzcd}$$

analysis here

$$\begin{tikzcd}
H^0(\text{top}) \arrow[r] & H^0(\text{bottom}) \arrow[r] & H^0(\text{bottom} - \Sigma_n^*) \\
\arrow[rr, bend right, "\text{iso}"] & &
\end{tikzcd}$$

but $H^0_{\Sigma_n}(\text{smooth, base}) = 0$

& then we actually get $H^1 = 0$ overleaf too.

See SGA7

$\mathcal{L}_n \to R\Psi_{\bar\eta}\mathcal{L}_n \to R\Phi_{\bar\eta}\mathcal{L}_n$ $\bar\eta = \operatorname{Spec}\overline{\mathbb{Q}}_p$

Complexes of sheaves.

A triangle. Get a long exact seq. Recall $\mathbb{H}^{i}(R\Phi_{\bar\eta}\mathcal{L}_n) = 0$ if $i \neq 1$.

$$0 \to H^1_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n) \xrightarrow{\sim} \overset{\mathbb{H}}{~~}^1_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R\Psi_{\bar\eta}\mathcal{L}_n)$$

$\overset{\shortparallel}{H^0}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{G}_n)$ see SGA7. RΨ doesn't [illegible] reduced why yet. related systems by [illegible] form.

$$\to H^0_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R^1\Phi_{\bar\eta}\mathcal{L}_n) \to H^2_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n)$$

$\shortparallel$

$(\mathbb{H}^1)$

$$\to \mathbb{H}^2_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R\Psi_{\bar\eta}\mathcal{L}_n) \to 0$$

$$H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{G}_n) \times H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R^1\Phi_{\bar\eta}\mathcal{L}_n)$$

$$\downarrow\wr$$

$$\mathbb{H}^1_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R\Psi_{\bar\eta}\mathcal{L}_n) \times H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R^1\Phi_{\bar\eta}\mathcal{L}_n)$$

Given a pairing $\mathcal{L}_n \times \mathcal{L}_n \to \mathbb{Z}_\ell$ ∃ cup product in hypercoh.

All in SGA7

$$\downarrow$$

$$\mathbb{H}^2_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R\Psi_{\bar\eta}\mathbb{Z}_\ell)$$

$$\downarrow \mathrm{tr}$$

$\underset{\text{u-Hy}}{\mathbb{Z}_\ell(-1)}$ (a perfect pairing)

So top 2 H^0's are dual.

This is a lot of structure.

Also $N : H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R^1\Phi_{\bar\eta}\mathcal{L}_n) \xrightarrow{\sim} \mathbb{H}^1_{\Sigma_N^*}(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R\Psi_{\bar\eta}\mathcal{L}_n)$

$\wr\shortparallel$

$H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{G}_n)$

In SGA7 it's called $\sum_{x \in \Sigma_N^*}(-, \delta_x)\delta_x(\text{?})$

So we have a duality & a direct iso

Recall $\sigma \in I_{\mathbb{Q}_p}$ acts as $1 + \operatorname{var}(\sigma)$, $\operatorname{var}(\sigma) = t_\ell(\sigma)N$

$t_\ell : I_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_\ell$

N commutes with $GL_2(\mathbb{A}^{\infty,p})$

$NF^* = pF^*N$

set $H^0(R^1\Phi\,\mathcal{L}_n) = \varinjlim_{N} H^0(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, R^1\Phi_{\bar{\eta}}\,\mathcal{L}_n(\mathbb{Z}_\ell))$

$$H^0(R^1\Phi\,\mathcal{L}_n)^- = \ker\left(H^0(R^1\Phi\,\mathcal{L}_n) \to \varinjlim_{N} H^2_c(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))\right)$$

$$\left(= H^0(R^1\Phi\,\mathcal{L}_n) \text{ if } n \neq 0 \right)$$

Fact: $N : H^0(R^1\Phi\,\mathcal{L}_n)^- \longrightarrow H^0(\mathcal{G}_{\underset{n}{N}})^-$ is iso.

He doesn't understand pf of this. Carayol gives a pf in "Hilbert mod forms"
He says ∃ pf in Langlands, Antwerp 2

Summing up, M'lud,

$$H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)/\overline{\mathbb{F}}_p)^+ = \varinjlim_{N} H^1_p(\mathcal{M}_{N,p} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))$$

Lemma

(Everything has an action of Frob.
(careful! F^*)

$$\begin{tikzcd}
 & 0 & & & \\
 & H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell)/\overline{\mathbb{F}}_p)^2 \arrow[u] & & & \\
0 \arrow[r] & H^1(\mathcal{L}_n(\mathbb{Z}_\ell)/\overline{\mathbb{F}}_p)^+ \arrow[r] \arrow[u] & H^1(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_0(p)} \arrow[r] & H^0(R^1\Phi\,\mathcal{L}_n(\mathbb{Z}_\ell))^- \arrow[r] \arrow[dll, "N \ \cong"'] & 0 \\
 & H^0(\mathcal{G}_n) \arrow[u] & & & \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

$\sigma \in T_{\mathbb{Q}_p}$ acts by $1 + t(\sigma)N$ on $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_0(p)}$

So we're gonna get $\rho_{\pi,\lambda}|_{D_p}$ if π_p special. Not difficult now.

Next time he'll try & explain why Rbt's this is true if $p \not\equiv 1\ (\ell)$.

(NT)

Wed 11/3/92

Recall

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & Y \arrow[d] & & & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & H^1_p(\mathcal{L}_n(\mathbb{Z}_l))^{U_0(p)} \arrow[r] & X \arrow[r] \arrow[ull, bend right, "N \cong"'] & 0 \\
 & \left(H^1_p(\mathcal{L}_n(\mathbb{Z}_l))^{GL_2(\mathbb{Z}_p)}\right)^2 \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

$U_0(p) \subseteq GL_2(\mathbb{Z}_p)$

$GL_2(\mathbb{A}^{\infty,p})$ acts on everything equivariantly
F^* acts on everything 'cept $H^1(\mathcal{L}_n(\mathbb{Z}_l))^{U_0(p)}$
$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $H^1(\mathcal{L}_n(\mathbb{Z}_l))^{U_0(p)}$ } I think.

The exact sequences commute with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ & F^* w.r.t. $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \ni F^* = \mathrm{Frob}_p^{-1}$

$E_1/\mathbb{F}_p$ s.s. I think

$I_{\mathbb{Q}_p}$ acts via $\sigma \mapsto 1 + t_l(\sigma)N$

$\mathbb{D}/\mathbb{Q}_p$ ~~[illegible]~~ quat alg, $S(\mathbb{D}) = \{p, \infty\}$, $\mathbb{D}^p = \mathbb{D} \cap \mathcal{O}_{\mathbb{D},p}$

$$0 \to \begin{cases} 0 & n \neq 0 \\ \mathrm{Map}((\mathbb{Q}^p)^\times \backslash (\mathbb{A}^{\infty,p})^\times, \mathbb{Z}_l), & n=0 \end{cases} \to \mathrm{Map}((\mathbb{D}^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), S^n(\mathbb{Z}_l^2)) \to Y \to 0$$

$GL_2(\mathbb{A}^{\infty,p})$ acts by $g(f) = g_l f(\cdot g)$; $F^*(f) = f(\pi^{-1}\cdot)$

A few rks about this situation:

$$\mathrm{Map}((\mathbb{D}^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), S^n(\mathbb{Z}_l^2))$$

$$\cong \mathrm{Map}(\mathbb{D}^\times \backslash \mathbb{D}^\times_{\mathbb{A}^\infty} / \mathcal{O}^\times_{\mathbb{D},p}, S^n(\mathbb{Z}_l^2))$$

($\exists$ natural map one way & easy to check it's iso)

$\mathrm{Map}\left((D^p)^\times \backslash GL_2(\mathbb{A}^{\infty,p}), S^n(\mathbb{Z}_\ell^2)\right) \otimes \mathbb{Q}_\ell$

$\cong \left\{ f : D^\times_{\mathbb{A}^\infty} / \mathcal{O}^\times_{D_p} \to S^n(\mathbb{Q}_\ell^2) \text{ locally ct} \;\middle|\; f(\delta\,\_) = \delta f(\_) \; \forall \delta \in D^\times \right\}$

via $\varphi \longmapsto (h \mapsto h_\ell \varphi(h))$

& $(h \mapsto h_\ell^{-1} f(h)) \longleftarrow f$

Now $GL_2(\mathbb{A}^{\infty,p})$ acts by g: $f \mapsto f(\_\, g)$ & $F^*(f) = -f(\pi_p\, \_)$

$\pi_p \in \mathcal{O}_{D_p}$ image of π

In ptic this is exactly the description of modular forms!!

So $\Rightarrow Y \otimes_{\mathbb{Z}_\ell} \mathbb{C} \cong \left(S^D_{n+2}\right)^{\mathcal{O}^\times_{D_p}} \Big/$ trivial elts if $n=0$.

$= \bigoplus \pi_i$, $\pi_i = \bigotimes_v \pi_{i,v}$

$\mathbb{Z}_\ell \hookrightarrow \mathbb{C}$ somehow

(unr) $\pi_{i,p}$ f.d. = a character

F^* acts on π_i by $(-)\,\pi_{i,p}(\pi_p)$

$\uparrow$

$\pi_p \in D^\times$

$\pi_i \leftrightarrow \pi_i'$ occurs in S_{n+2} (Jacquet Langlands)

Check $\pi_{i,p}(\pi) = \pi_i'(U_p)$

__Cor__ (Implication of sequences on Galois rep's of π or sthg)

If π is an irred admiss rep of $GL_2(\mathbb{A}^\infty)$ occurring in $H^1_p(\mathcal{L}_n(\mathbb{Q}))$ & if $\pi_p \cong S(\chi, \chi|\cdot|_p)$ with χ unr & if λ is a prime of E_π of residue char $\neq p$ then

$$\rho_{\pi,\lambda}\Big|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_1 \chi_\ell^{-1} \end{pmatrix}$$

where $*$ is non-trivial & $\chi_1(\mathrm{Frob}_p^{-1}) = \pi_p(T_p)$

$(\omega \neq 0)$

Pf Immediate from what we had before. □ (see small writing above!)

FACT (Carayol) ∀ ~~pt~~ $p \neq$ res char of λ then

$$\mathrm{cond}\left(\rho_{\pi,\lambda}\Big|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\right) = \text{cond of } \pi_p$$

$= $ highest power of p dividing N where N is a sort of global conductor:
ie N is min^l s.t. $\pi^{U_1(N)} \neq 0$
(i.e. π has a fixed vector by $V_1(N)$)

Finite levels may not preserve exactness take invariants ~~of~~ from diagram anyway true

ie Rk The same statements remain true at finite level U if U small enough, e.g. $U \subseteq U_1(N)$, $N \geq 3$.

ie

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & Y_U \arrow[d] & & & \\
0 \arrow[r] & Z_U \arrow[r] \arrow[d] & H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_0(p)\times U} \arrow[r] & X_U \arrow[r] \arrow[ull, bend right, "N", "\cong"'] & 0 \\
 & \left(H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{GL_2(\mathbb{Z}_p)\times U}\right)^{t} \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

$U \subseteq GL_2(\mathbb{A}^{\infty,p})$
open comp subgp

Hecke equiv.
Galois equiv

$$0 \to \begin{Bmatrix} 0 & \text{if } n>0 \\ \mathrm{Map}(\mathbb{Q}^\times\backslash\mathbb{A}^{\infty\times}/\mathbb{Z}_p^\times, \mathbb{Z}_\ell) & n=0 \end{Bmatrix} \to \mathrm{Map}\left(D^\times\backslash D^\times_{\mathbb{A}^\infty}/\mathcal{O}^\times_{D,p}, S^n(\mathbb{Z}_\ell^2)\right) \to Y_U \to 0$$

End of alg geom

4. Congruences

It's not at all clear how to do this adelically if indeed its sensible to.

Let $U = U_1(N) \cap U_0(M) \subseteq GL_2(\mathbb{A}^\infty)$

Let $h_k(U) = \mathbb{Z}$-alg in $\mathrm{End}_{\mathbb{C}}(S_k^U)$ gen by $T_p = [U\begin{pmatrix}1&0\\0&\pi_p\end{pmatrix}U]$
$S_p = [U\begin{pmatrix}\pi_p&0\\0&\pi_p\end{pmatrix}U]$ $p\nmid N$

Assume $k \geq 2$.

Lemma 1) $h_k(U) = \mathbb{Z}$-alg gen by T_p & S_p in $\mathrm{End}_{\mathbb{C}}(H^1_p(\mathcal{L}_{k-2}(\mathbb{C}))^U)$ $p\nmid NM$ $e\nmid M$

$= \mathbb{Z}$-alg gen by T_p & S_p in $\mathrm{End}_{\mathbb{Q}}(H^1_p(\mathcal{L}_{k-2}(\mathbb{Q}))^U)$

$= \mathbb{Z}$-alg gen by T_p & S_p in $\mathrm{End}_{\mathbb{Z}_\ell}(H^1_p(\mathcal{L}_{k-2}(\mathbb{Z}_\ell))^U)$ $\forall \ell$

2) $h_k(U)$ is f.g as a $\mathbb{Z}$-module

Pf Crucial pt is 1st statement in 1.

1) Use $H^1_p(\mathcal{L}_{k-2}(\mathbb{C})) \cong S_k \oplus \overline{S_k} \cong S_k \oplus S_k$ (if π occurs then $\overline{\pi}$ occurs)

→ 1st line.

$\mathrm{End}_{\mathbb{Q}} \otimes \mathbb{C} = \mathrm{End}_{\mathbb{C}}$ & T_p, S_p ~~preserve~~ $\in \mathrm{End}_{\mathbb{Q}}(H^1_p(\mathcal{L}_n(\mathbb{Q})))$

$$\cap$$

$\mathrm{End}_{\mathbb{Q}}(\text{——}) \otimes \mathbb{C}$

$$\parallel$$

$\mathrm{End}_{\mathbb{C}}(\text{——}\ \mathbb{C})$

3rd eq same pick $\mathbb{Z}_p \hookrightarrow \mathbb{C}$

1) ⇒ 2). Exercise □

<u>Lemma</u> If $\theta : h_k(U) \to \mathbb{C}$ then $\exists ! \, \pi_\theta$ cusp. auto. rep of $GL_2(\mathbb{A}^\infty)$ of wt k s.t $\forall p \nmid NM$ $\pi_{\theta,p}$ unr & $\pi_{\theta,p}\begin{Bmatrix} T_p \\ S_p \end{Bmatrix} = \theta\begin{Bmatrix} T_p \\ S_p \end{Bmatrix}$

<u>Pf</u> Easy given cusp forms are direct sum of irreds. Exercise □

Say $\overset{U}{\theta} \sim \overset{U'}{\theta'}$ if $\theta\binom{T_p}{S_p} = \theta'\binom{T_p}{S_p}$ for all but finitely many p

<u>Cor</u> $\exists$ bij between $\begin{Bmatrix} \text{equiv classes} \\ \text{of } \theta \end{Bmatrix}$ & $\begin{Bmatrix} \text{cusp auto} \\ \text{~~rep~~ of wt } k \end{Bmatrix}$

$$\theta \longmapsto \pi_\theta$$

□

<u>Exercise</u> If $[\theta]$ is a $\sim$ class, $\exists N$ min^l s.t $\exists \theta_1 \in [\theta]$, $\theta_1 \in h_k(U_1(N))$ & if $\theta \in [\theta]$ & $\theta : h_k(U_1(M)) \to \mathbb{C}$ then $N | M$

We call N the <u>conductor</u> of $[\theta]$

<u>Cor</u> Given θ as above, $\mathbb{Q}(\operatorname{Im}\theta)$ is a number field $=_{df} E_\theta$

$\theta : h_k(U) \to \mathcal{O}_{E_\theta}$

& ~~rep~~ for each prime λ of $\mathcal{O}_{E_\theta}$ $\exists$ $\rho_{\theta,\lambda} : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathcal{O}_{E_{\theta,\lambda}})$

s.t 1) $\rho\ell$ $\rho_{\theta,\lambda}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2) If $p \nmid NM$ & $p \neq$ res char λ then $\rho_{\theta,\lambda}$ unr at p & $\rho_{\theta,\lambda}(\mathrm{Frob}_p^{-1})$ has char poly $X^2 - \theta(T_p)X + p\theta(S_p)$

3) ~~cond~~ If $p \neq$ res char λ then cond $\rho_{\theta,\lambda}|_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ = highest pow of p in cond(θ)

4) If $U = U_1(N) \cap U_0(p)$, $p \nmid N$, $p \neq$ res char λ, $p |$ cond(θ), then $\pi_{\theta,p}$ is special ie $S(\chi, \chi|\cdot|_p)$.

& $\rho_{\theta,\lambda}|_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_1\chi_\ell^{-1} \end{pmatrix}$ where $\underset{\& \, * \neq 0}{}$ $\chi_1 : Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to E_{\theta,\lambda}^\times$ unr & $\chi_1(\mathrm{Frob}_p^{-1}) = \theta(T_p)$

Pf Ex (limit of finite level n stly) □

One way of thinking about modular forms mod p is to think about ~~mod form~~ HMs of Hecke algebra with values in a finite field.

So we'll now consider HMs $\theta : h_k(U) \to \overline{\mathbb{F}}_\ell$ NB $h_k(U)$ is commutative

Lemma If $\theta : h_k(U) \to \overline{\mathbb{F}}_\ell$ then $\exists \tilde{\theta}$ $h_k(U) \to \mathbb{C}$ & a prime λ of $\mathcal{O}_{E_{\tilde{\theta}}}$ s.t.

$$\begin{tikzcd} & h_k(U) \arrow[dl, "\theta"'] \arrow[dr, "\tilde{\sigma}"] & \\ \overline{\mathbb{F}}_\ell & & \mathcal{O}_{E_{\tilde{\theta}}} \arrow[ll, "\text{reduce}"', "\text{mod } \lambda"] \end{tikzcd}$$

commutes

(he's not really thinking about wt 1 but this is true in wt 1)

Pf Let $\mathfrak{m} = \ker\theta$, a ~~not~~ max^l ideal of $h =_{df} h_k(U)$

$\mathfrak{m} \cap \mathbb{Z} = (\ell)$ $h_{\mathfrak{m}} \ni \ell$ & ℓ not nilpotent

(as if it were, ℓ's $= 0$ in h for some $s \notin \mathfrak{m}$ but h is a free $\mathbb{Z}$-module)

(torsion)

$\Rightarrow \exists \mathfrak{p}'$, a prime of $h_{\mathfrak{m}}$ s.t. $\mathfrak{p}' \not\ni \ell$ (look it up in a book)

Hence $\exists$ prime $\mathfrak{p}$ of h s.t. $\mathfrak{p} \subset \mathfrak{m}$, $\ell \notin \mathfrak{p}$.

Then $\mathfrak{p} \cap \mathbb{Z}$ is a prime of $\mathbb{Z}$, not containing ℓ, so its zero.

$\tilde{\theta} : h \to h/\mathfrak{p}$ $h/\mathfrak{p}$ contains $\mathbb{Z}$ & is an ID so embeds in $\overline{\mathbb{Q}}$.

(field of fractions)

(⇐ Going down thm) (B/A flat ⇒ $A \to B$, $\mathfrak{p} \subset A$ $\exists \mathfrak{p}'$ of B $\mathfrak{p}' \cap A = \mathfrak{p}$, $\mathfrak{p}' \subset \beta$)

<u>Cor</u> If $\theta : h_k(U) \to \overline{\mathbb{F}}_\ell$, $(k \geq 2)$ then $\exists$ cts rep $\rho_\theta : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_\ell)$ s.t.

1) ~~all~~ $\rho_\theta(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2) if $p \nmid \ell MN$ then ρ_θ unr @ p & $\rho_\theta(\mathrm{Frob}_p^{-1})$ has char poly $X^2 - \theta(T_p)X + p\,\theta(S_p)$

3) $\mathrm{cond}\, \rho_\theta \mid NM\ell^\infty$

note no equality. see thm of Mazur below. see cong too.

4) $\theta \sim \theta' \Leftrightarrow_{\mathrm{df}} \theta\binom{T_p}{S_p} = \theta'\binom{T_p}{S_p}$ ~~for~~ all but finitely many p

$\theta \sim \theta' \Rightarrow \rho_\theta = \rho_{\theta'}$

5) If $\exists$ a lift $\tilde{\theta}$ of θ st $(\pi_{\tilde{\theta}})_p = S(\chi, \chi|\cdot|_p)$, χ unr, then

$$\rho_\theta\big|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_1 \chi_\ell^{-1} \end{pmatrix} \text{ where}$$

$\chi_1 : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \overline{\mathbb{F}}_\ell$ is unr & $\chi_1(\mathrm{Frob}_p^{-1}) = \theta(T_p)$

(if $p \| NM$)

~~THM (Mazur) Suppose θ: $h_k(U_1(N))$~~

<u>Conj</u> Given θ st ρ_θ (abs) irred $\exists \theta' \sim \theta$, θ' on $h_k(U_1(\mathrm{cond}\, \rho_\theta))$

(ie the same happens as in char 0)

1st step in proving this, a hard thm is

<u>THM</u> (Mazur) Suppose $\theta : h_k(U_1(N) \cap U_0(p)) \to \overline{\mathbb{F}}_\ell$ where $p \nmid N\ell$, $N \geq 3$, $k \geq 2$
& suppose ρ_θ irred
& unr at p.
Assume further that $p \not\equiv 1 (\ell)$. (This condn will be removed by Ribet)
Then $\exists \theta' \sim \theta$, $\theta' : h_k(U_1(N)) \to \overline{\mathbb{F}}_\ell$

This (& implies) is implied trivially by the following restatement (contrapositive)

THM' Suppose $\theta : h_k(U_1(N) \cap U_0(p)) \to \overline{\mathbb{F}}_\ell$, $p \nmid N\ell$, $N \geq 3$, $k \geq 2$, $p \not\equiv 1 (\ell)$.
Suppose ρ_θ irred & $\nexists \theta' \sim \theta$, $\theta' : h_k(U_1(N)) \to \overline{\mathbb{F}}_\ell$
Then ρ_θ is ramified at p

Pf Use that ludicrous diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & Y_\theta \arrow[d] & & \\
0 \arrow[r] & Z_\theta \arrow[r] \arrow[d] & H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_1(N) \cap U_0(p)} \arrow[r] & X_\theta \arrow[r] \arrow[lllu, bend right, "N", "\cong"'] & 0 \\
 & (H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_1(N)})^2 \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

Let $h = h_k(U_1(N) \cap U_0(p))$, $\mathfrak{m} = \ker \theta$

Because $\nexists \theta' \sim \theta$, $\theta' : h_k(U_1(N)) \to \overline{\mathbb{F}}_\ell$,

we have $\left(H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_1(N)} \right)^2_{\mathfrak{m}} = (0)$ (as $\nexists$ eigenf. here)

Localisation is exact $\therefore$

$$\begin{tikzcd}
0 \arrow[r] & Y_{\mathfrak{m}} \arrow[r] & H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))^{U_1(N) \cap U_0(p)}_{\mathfrak{m}} \arrow[r] & X_{\mathfrak{m}} \arrow[r] \arrow[ll, bend left=60, "N"] & 0 \\
 & & H_{\mathfrak{m}} \arrow[u, equal, "\text{def}"'] & &
\end{tikzcd}$$

Now $\otimes_h h/m$

$$\begin{tikzcd} Y\otimes_h h/m \arrow[r] & H\otimes_h h/m \arrow[r] & X\otimes_h h/m \arrow[r] \arrow[ll, bend right, "N", "\sim"'] & 0 \end{tikzcd}$$

$\parallel$ (0) as h acts faithfully on H.

Lemma $H\otimes_h h/m \cong \rho_\theta^{a}$ (ie some power)

Pf **Step 1** $(H\otimes_h h/m)^{ss} \cong \rho_\theta^{a}$

- suppose ρ is a JH cpt of LHS: $\rho \oplus (\det\rho_\theta)\check{\rho} \cong \rho_\theta^{d}$
as char polys of Frob_p^{-1} are same for almost all p

why: RHS we understood.
LHS: $\rho(\mathrm{Frob}_p^{-1})$ satisfies $X^2 - \theta(T_p)X + p\theta(S_p) = 0$
$\uparrow$ [illegible]

- say this has roots α_p, β_p

Then $\rho(\mathrm{Frob}_p^{-1})$ has evals α_p mult r, β_p mult $\dim\rho - r$

$\therefore$ $\det\rho_\theta\,\check{\rho}(\mathrm{Frob}_p^{-1})$ has evals β_p mult r, α_p mult $\dim\rho - r$ ✓

Step 2 $H\otimes_h h/m$ is ss.

Pf is easy but he hasn't looked at it yet.
It's because $(H\otimes_h h/m)^{ss} \cong \rho_\theta^{c}$, ρ_θ 2dim'l abs irred
& $\mathrm{char}_{\rho_\theta(g)}(\rho_{H\otimes_h h/m}(g)) = 0\ \forall g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
He'll explain next time. $\square$

Cor If ρ_θ were unr @ p, then $H\otimes_h h/m$ is unr @ p

But
$$\begin{tikzcd} Y\otimes_h h/m \arrow[r] & H\otimes_h h/m \arrow[r] & X\otimes_h h/m \arrow[r] \arrow[ll, bend left, "N"', "\sim"] & 0 \end{tikzcd}$$
& $I_{\mathbb{Q}_p}$ acts by $1+t_\ell(\sigma)N$,
$t_\ell : I_{\mathbb{Q}_p} \to \mathbb{Z}_\ell$

$$\begin{tikzcd} Y \otimes_h h/m \arrow[r, "A"] & H \otimes_h h/m \arrow[r, "B"] & X \otimes_h h/m \arrow[r] \arrow[ll, bend right, "N"'] & 0 \end{tikzcd}$$

<u>Cor</u> ρ_θ unr @p $\Rightarrow$ $ANB = 0 \Rightarrow A = 0$

<u>Cor</u> ρ_θ unr @p $\Rightarrow$ $H \otimes h/m \xrightarrow{\sim} X \otimes h/m$

But this map preserves the action of Galois.
So $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts by scalars on $H \otimes h/m$

$\Rightarrow$ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts by scalas on ρ_θ

But $\rho_\theta(\sigma) \sim \begin{pmatrix} \alpha & * \\ 0 & \alpha p \end{pmatrix}$ $\quad \sigma \mapsto \mathrm{Frob}_p^{-1}$

& $p \not\equiv 1 \ (\ell) \Rightarrow$ not a scalar. #. $\square$

Lecture 19, copied off Kanten who copied it off Frazer

Thm 1 (Ribet) Suppose $\theta : h_k(U_1(N) \cap U_0(p)) \to \overline{\mathbb{F}}_\ell$ where $p \nmid N\ell$, and $N \geq 3$, $\ell \neq 2$ ($k \geq 2$ as usual) Suppose $\rho_\theta : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_\ell)$ is irred, & unr. at p. Then $\exists \theta' : h_k(U_1(N)) \to \overline{\mathbb{F}}_\ell$ s.t. $\rho_\theta = \rho_{\theta'}$.

Recall we proved this before when $p \not\equiv 1(\ell)$ - Mazur's thm.

Thm 2 (Ribet) Suppose $\theta : h_k(U_1(N) \cap U_0(pq)) \to \overline{\mathbb{F}}_\ell$ where $p \nmid N\ell$, $q \nmid N\ell$, $p \neq q$, $\ell \neq 2$, $N \geq 3$ Suppose ρ_θ is irred. & unr at p. If also $q \not\equiv 1 \ (\ell)$ & θ ~~cor~~ factorises via $h_k(U_1(N) \cap U_0(pq))^{q\text{-new}}$ then $\exists \theta' : h_k(U_1(N) \cap U_0(q)) \to \overline{\mathbb{F}}_\ell$ with $\rho_\theta = \rho_{\theta'}$.
Here, $h_k(U_1(N) \cap U_0(pq))^{q\text{-new}}$ is defined by
$S_k(U_1(N) \cap U_0(pq)) = S_k(U_1(N) \cap U_0(p))^2 \oplus S_k(U_1(N) \cap U_0(pq))^{q\text{-new}}$. The Hecke operators preserve this decomposition, so $h_k(U_1(N) \cap U_0(pq)) =$ ~~$S_k(U_1(N) \cap U_0(p))^2$~~ image of $h_k(U_1(N) \cap U_0(pq))$ in $\mathrm{End}_{\mathbb{C}}(S_k(U(N) \cap U_0(pq))^{q\text{-new}}$.

~~Note~~

Note that thm 2 ⇒ thm 1: Given θ as in thm 1, find $q \not\equiv 1(\ell)$ s.t. $\theta(T_q)^2 = \theta(S_q)(q+1)^2$, $q \nmid N\ell p$ eg $\rho_\theta(\mathrm{Frob}_q) \sim \rho_\theta(c)$
$\uparrow$ level $U_1(N) \cap U_0(p)$

(then $q \equiv -1(\ell)$, $\theta(T_q) = \theta(S_q) = 0$ - can find such q using Cebotarev)

Then there's a thm of Diamond which implies that θ extends to $\tilde{\theta} : h_k(U_1(N) \cap U_0(pq))^{\ell} \to \overline{\mathbb{F}}_\ell$ & $\rho_{\tilde{\theta}} = \rho_\theta$, so that $\tilde{\theta}$ factors through the q-new part

By thm 2, $\exists \theta' : h_k(U_1(N) \cap U_0(q)) \to \overline{\mathbb{F}}_\ell$ with $\rho_{\theta'} = \rho_\theta$ unramified at q. Now, as $q \not\equiv 1(\ell)$, apply Mazur's thm: $\exists \theta'' : h_k(U_1(N))^{\ell} \to \overline{\mathbb{F}}_\ell$ s.t. $\rho_{\theta''} = \rho_{\theta'} (= \rho_{\tilde{\theta}})$

(Diamond's thm is essentially elementary, given that
$H^1_p(\mathcal{M}_{U_1(N)}, \mathcal{L}_n(\overline{\mathbb{F}}_\ell))^2 \hookrightarrow H^1_p(\mathcal{M}_{U_1(N) \cap U_0(q)}, \mathcal{L}_n(\overline{\mathbb{F}}_\ell))$)

Proof of thm 2: Let $B/\mathbb{Q}$ be the quaternion algebra over $\mathbb{Q}$ ramified at exactly $\{p,q\}$. Given $U \subseteq GL_2(\mathbb{A}^{pq}_f)$ open, cpct, s.t. $U \subseteq U_1(N)$, for some $N \geq 3$, level N ($U \supseteq U_N$)

Then we get a Shimura curve $\mathfrak{X}_U / \operatorname{spec} \mathbb{Z}[1/N]$

- $\mathfrak{X}_U$ is a moduli space of certain abelian surfaces.
$\mathfrak{X}_U$ is proper, though not nec. smooth.
It's smooth over spec $\mathbb{Z}[1/Npq]$

Have ℓ-adic sheaves $\mathcal{L}_n(\mathbb{Z}_\ell)$ on $\mathfrak{X}_U$ for $\ell \neq p,q$

$$0 \to S^B_{n+2}(\tilde{U}) \to H^1(\mathfrak{X}_U(\mathbb{C}), \mathcal{L}_n(\mathbb{C})) \longrightarrow \underset{\text{(split)}}{\overline{S^B_{n+2}(\tilde{U})}} \to 0$$

Here $\tilde{U} = U \times \mathcal{O}^\times_{B,p} \times \mathcal{O}^\times_{B,q}$

$$H^1(\mathfrak{X}_U \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Q}_\ell)) = \bigoplus_{\pi} (\pi^{pq})^U \underset{E_\pi}{\otimes} \Big(\bigoplus_{\lambda | \ell} \rho_{\pi,\lambda} \Big)$$

(π runs over...

π runs over irred. admissible rep's of $GL_2(\mathbb{A}_f)$ in $H^1_p(\mathcal{L}_n(\mathbb{Z}_\ell))$ s.t. π_p, π_q are special ($\leftrightarrow$ unr. chars)

As before we have an exact sequence

(1)
$$\begin{tikzcd} 0 \arrow[r] & X_p(B,U,n) \arrow[r] & H^1(\mathcal{X}_U \times \overline{\mathbb{Q}}_p, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[r] & X_p(B,U,n)^* \arrow[r] \arrow[ll, bend left, "N_{B_\ell}", "/" marking] & 0 \end{tikzcd}$$
,

equivariant for $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ & the Hecke algebra
$\sigma \in I_{\mathbb{Q}_p}$ acts by $1 + t_\ell(\sigma) N$, $t_\ell : I_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_\ell$

Warning: N is no longer surjective (the Shimura curve has singularities; blowing up adds copies of $\mathbb{P}^1$ - N is not surjective onto these $\mathbb{P}^1$'s)

Recall
$$\begin{tikzcd} 0 \arrow[r] & X_q(V,n) \arrow[r] & H^1_p(\mathcal{M}_{V \times U_0(q)} \times \overline{\mathbb{Q}}_\ell, \mathcal{L}_n(\mathbb{Z}_\ell)) \arrow[r] & X_q(V,n)^* \arrow[r] \arrow[ll, bend left, "\sim", "N_V"'] & 0 \end{tikzcd}$$

$V \subseteq GL_2(\mathbb{A}^q_f)$
To compute $X_p(B,U,n)$ we get

(2)
$$0 \to X_q(U \times GL_2(\mathbb{Z}_p), n)^2 \to X_q(U \times U_0(p), n) \to X_p(B, U, n) \to 0$$
(Hecke-equivariant)

Motivation for this short exact sequence.

$$\begin{tikzcd} S^{D_q^\infty}_{n+2}(U \times GL_2(\mathbb{Z}_p))^2 \arrow[r] & S^{D_q^\infty}_{n+2}(U \times U_0(p)) \arrow[r, "\text{quotient}"] & S^{D_q^\infty}_{n+2}(U \times U_0(p))^{p\text{-new}} \arrow[d, leftrightarrow, "\text{Jacquet Langlands}"] \\ & & S^{D_{pq}}_{n+2}(U) \end{tikzcd}$$

$$N_{B,U} = N_{U\times U_0(p)}\Big|_{X_p(B,U,n)^*}$$

May reverse rôles of p,q - we haven't yet used any specific properties of p,q as in Thm2.

Define $\Phi_p(B,n) = X_p(B,U,n) \Big/ N_{B,U} X_p^*(B,U,n)^*$

$$= X_q(U\times GL_2(\mathbb{Z}_p),n) \Big/ \left(N^{-1}_{U\times U_0(p)} X_q(U\times GL_2(\mathbb{Z}_p),n)\right)^2 \quad \text{by(2)}$$

[illegible] (exercise)

$$= X_q(U\times GL_2(\mathbb{Z}_p),n)^{*2} \Big/ \underbrace{N^{-1}_{U\times U_0(p)} N_{U\times GL_2(\mathbb{Z}_p)}}_{\eta_p} X_q(U\times GL_2(\mathbb{Z}_p),n)^{*2}$$

Fact: $\eta_p = T_p^2 - S_p$

$\uparrow$ Level $U\times U_0(p)$

Can still swap p&q - so in (1):

(3)

$$\begin{tikzcd}
0 \arrow[r] & X_p(B,U,n) \arrow[r] \arrow[d] & H^1(X_r\times\bar{\mathbb{Q}}_p, \mathcal{L}_n(\mathbb{Z}_l)) \arrow[r] & X_p(B,U,n)^* \arrow[r] \arrow[ll, bend left, "N_{B,U}"] & 0 \\
& X_q(U\times GL_2(\mathbb{Z}_p),n)^{*2} / T_p^2 - S_p \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}$$

(can reverse p,q)

Return to proving Thm 2.

$$H^1(\mathcal{X}_{U_1(N)} \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell)) \otimes_{h} h/\mathfrak{m} \cong V^{\mu}$$

$$H^1_p(\mathcal{M}_{U_1(N) \cap U_0(pq)} \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell)) \otimes_{h} h/\mathfrak{m} \cong V^{\lambda}, \text{ by BLR lemma}$$

$(V = \rho_\theta,\ \mathfrak{m} = \ker\theta,\ h = h_{n+2}(U_1(N) \cap U_0(pq)))$

Suppose θ' does not exist.

$$\mathcal{X}_{U_1(N)} \bmod p, \quad \mathcal{M}_{U_0(pq) \cap U_1(N)} \bmod q$$

Observe $\Phi_p(B,n)_{\mathfrak{m}} = (0)$ else in (3) $X_q(U \times GL_2(\mathbb{Z}_p), n)^{*2}_{\mathfrak{m}} \neq (0)$ and $H^1_p(\mathcal{M}_{U_1(N) \times U_0(pq)} \times \overline{\mathbb{Q}}, \mathcal{L}_n(\mathbb{Z}_\ell))^2_{\mathfrak{m}} \neq (0)$ so θ' exists.

Tensor (3) with $h/\mathfrak{m}$:

$$\begin{tikzcd} X_p(B,U_1(N),n) \otimes h/\mathfrak{m} \arrow[r, "\text{must be zero map}"'] & V|^{\mu}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \arrow[r] & X_p(B,U_1(N),n)^* \otimes h/\mathfrak{m} \arrow[r] \arrow[ll, bend left, "N_{q,U_1(N)}"] & 0 \end{tikzcd} \quad (4)$$

$$\Rightarrow V|^{\mu}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong X_p(B,U_1(N),n)^* \otimes h/\mathfrak{m}$$

As above, $X_q(U_1(N) \times GL_2(\mathbb{Z}_p), n)^{*2}_{\mathfrak{m}} = (0)$, and so

$$\begin{tikzcd} X_p(B,U_1(N),n)^* \otimes h/\mathfrak{m} \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr\wr"] & X_q(U(N) \times U_0(p), n)^* \otimes h/\mathfrak{m} \\ V|^{\mu}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} & \end{tikzcd}$$

(5)

$$\begin{tikzcd}
 & ? \arrow[d] & & & \\
 & X_q(U_1(N)\times U_0(p),n)\otimes h/m \arrow[d] & & & \\
? \arrow[r] & ? \arrow[r] \arrow[d] & V|^{\lambda}_{\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)} \arrow[r] & X_q(U_1(N)\times U_0(p),n)^{*}\otimes h/m \arrow[r] \arrow[ull, bend right, "\sim"', "N_{U_1(N)\times U_0(p)}"] \arrow[dll, "/" marking] & 0 \\
 & H^1(\mathcal{M}_{U_1(N)\times U_0(p)}\times\bar{\mathbb{F}}_q, \mathcal{L}_a(\mathbb{Z}_\ell))^2\otimes h/m \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

But on $V|_{\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)}$, Frob$_q$ has 2 distinct eigenvalues (as $q \not\equiv 1(\ell)$), each with multiplicity λ;

& on $X_q(U_1(N)\times U_0(p),n)^*$, Frob$_q$ has only 1 eigenvalue

Thus $\dim X_q(U_1(N)\times U_0(p),n)^*\otimes h/m \leq \lambda$

$\Rightarrow 2\mu \leq \lambda$

Now reverse p,q

$\mathcal{X}_{U_1(N)}$ mod q, $\mathcal{M}_{U_0(pq)\cap U_1(N)}$ mod p

As before (4):

$$\begin{tikzcd}
0 \arrow[r] & X_q(B,U_1(N),n)\otimes h/m \arrow[r] & V|^{\mu}_{\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)} \arrow[r] & X_q(B,U_1(N),n)^*\otimes h/m \arrow[r] \arrow[ll, bend left, "N"] & 0
\end{tikzcd}$$

(localise at $\mathfrak{m}$ & split into the 2 eigenspaces for Frobenius at q)
N must be 0 as V unr. at q.

$\Rightarrow X_q(B, U_1(N), n) \otimes h/\mathfrak{m} \cong$ cokernel

$$= \Phi_q(U_1(N), n) \otimes h/\mathfrak{m}$$

$$= X_p(U_1(N) \times GL_2(\mathbb{Z}_q), n)^{*2} \otimes h/\mathfrak{m} \Big/ (T_q^2 - S_q)$$

But $T_q^2 - S_q \in \mathfrak{M}$ as θ factors through the q-new part.

Thus $\Phi_q(U_1(N), n) \otimes h/\mathfrak{m} \cong X_p(U_1(N) \times GL_2(\mathbb{Z}_q), n)^{*2} \otimes h/\mathfrak{m}$

$$\Rightarrow \dim(X_p(U_1(N) \times GL_2(\mathbb{Z}_q), n)^{*2} \otimes h/\mathfrak{m}) = \underbrace{\dim(X_q(B, U_1(N), n)^* \otimes h/\mathfrak{m})}_{=\mu}$$

(as the 2 eigenspaces of Frob_q on V^μ have dim μ, as $q \not\equiv 1 (\ell)$)

But from (2)

$$? \to \underbrace{X_q(B, U_1(N), n)^* \otimes h/\mathfrak{m}}_{\dim = \mu} \to X_p(U_1(N) \times U_0(q), n)^* \otimes h/\mathfrak{m}$$

$$\to X_p(U_1(N) \times GL_2(\mathbb{Z}_q), n)^{*2} \otimes h/\mathfrak{m} \to 0$$

$\Rightarrow \dim X_p(U_1(N) \times U_0(q), n)^* \otimes h/\mathfrak{m} \leq 2\mu$

As in (5) get

$$\begin{tikzcd}
& \bullet \arrow[d] & & & \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet \arrow[r] & 0 \\
& \bullet \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}$$

& so $H^1_p(\mathcal{M}_{U_1(N) \times U_0(q)} \times \overline{\mathbb{F}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))_\mathfrak{m} = (0)$ as θ' does not exist.

So

$$\begin{tikzcd}
0 \arrow[r] & X_p(U_1(N) \times U_0(q), n)_\mathfrak{m} \arrow[r] & H^1_p(\mathcal{M}_{U_1(N) \times U_0(q)} \times \overline{\mathbb{Q}}_p, \mathcal{L}_n(\mathbb{Z}_\ell))_\mathfrak{m} \arrow[r] & X_p(U_1(N) \times U_0(q), n)^*_\mathfrak{m} \arrow[r] \arrow[lll, bend left, "N"] & 0
\end{tikzcd}$$

Tensoring with h/m:

$$\begin{tikzcd} X_p(U_1(N)\times U_0(q),n)\otimes h/m \arrow[r, "\alpha"] & V|^{\lambda}_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \arrow[r, "\beta"] & X_p(U_1(N)\times U_0(q),n)^*\otimes h/m \arrow[r] \arrow[ll, bend left, "\sim", "N"'] & 0 \end{tikzcd}$$

V unr. at p so $\alpha=0$ & β is an iso.

This $\dim V|^{\lambda}_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \leq 2\mu$

$\Rightarrow 2\lambda \leq 2\mu$

But $2\mu \leq \lambda \Rightarrow \lambda \leq 0$

But θ exists, & the Hecke action is faithful so $\lambda \geq 1$ #

□

(BOSTON - LENSTRA - RIBET)

<u>Lemma</u> Let G be a finite group and k a field and $\rho : G \longrightarrow Gl_2(k)$ an absolutely irred rep. Let W be a k vector space and $\sigma : G \longrightarrow \mathrm{Aut}(W)$ st $\forall g \in G$

$$\mathrm{char}_{\rho(g)}(\sigma(g)) = 0 .$$

Then $\sigma \cong \rho^d$.

<u>Pf</u> Let J be the 2-sided ideal of $k[G]$ generated by $g^2 - (\mathrm{tr}\,\rho(g))\, g + \det g \quad \forall g \in G$ and let $R = k[G]/J$. Then $\rho : R \longrightarrow M_2(k)$ and $\sigma : R \longrightarrow \mathrm{End}(W)$. We claim that ρ is an isomorphism from which the result will follow. It is surjective as ρ is absolutely irreducible. We must prove it injective.

There is an involution $* : k[G] \to k[G]$

$$g \longmapsto (\det \rho(g))\, g^{-1} .$$

$J^* = J$ so $* : R \longrightarrow R$.

$\forall x \in R \qquad x + x^* = \mathrm{tr}\,\rho(x)$ as it is true for $g \in$ image of G + is a linear relation

$\forall x \in R \qquad x x^* = \det \rho(x)$ as it is true for $g \in$ image of G and

$\mathfrak{X} = \{ x \in R \mid x x^* = \det \rho(x) \}$ is stable under addition

as $(x+y)(x+y)^* = \det \rho(x) + \det \rho(y) + \mathrm{tr}\,\rho(x y^*)$

$= \det\big(\rho(x) + \rho(y)\big) \quad \forall x, y \in \mathfrak{X}$

Now suppose $z \in \ker \rho$.

Then $\forall x \in R$ $\mathrm{tr}\,\rho z = \mathrm{tr}\,\rho(zx) = 0$ so $zx = -z^* x = x^* z$.

Thus $\forall x, y \in R \qquad z x^* y^* = y x z = x z y^* = x y z$ i.e. $(xy - yx) z = 0$.

Let $I = \{ x \in R \mid x z = 0 \}$. Then by the above equation we see that I is a 2-sided ideal of R. Thus $\rho I = (0)$ or $M_2(k)$ but $\rho I \supset [M_2(k), M_2(k)] \neq (0)$ so $\rho I = M_2(k)$.

Thus $\exists x \in R$ st $1 = \rho x$ and $x z = 0$. Then $x^* x z = z = 0$, and the result follows.